

Pseudofermion scattering theory

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In this paper we study the scattering theory associated with the pseudofermion dynamical theory for the Hubbard chain. While for the electronic basis the problem is non perturbative and strongly correlated, in terms of pseudofermions the spectral properties are controlled by zero-momentum forward scattering only. Indeed, we find that each ground-state – excited-energy-eigenstate transition corresponds to a well defined set of elementary two-pseudofermion zero-momentum forward-scattering events. An important point of the theory is that independent η -spin 1/2 holons and spin 1/2 spinons are neither scatterers nor scattering centers. Instead, the scatterers and scattering centers are spin-less and η -spin-less $c0$ pseudofermions, η -spin-zero 2ν -holon composite $c\nu$ pseudofermions, spin-zero 2ν -spinon composite $s\nu$ pseudofermions such that $\nu = 1, 2, 3, \dots$, and the corresponding pseudofermion holes. Similarly to chromodynamics, where all quark-composite physical particles are color-neutral, for the pseudofermion dynamical theory all 2ν -holon (and 2ν -spinon) composite pseudofermion scatterers and scattering centers are η -spin-neutral (and spin-neutral). Thus, the pseudofermion S matrix is a mere phase factor, which is behind the simple form of the pseudofermion anti-commutators and the simplification of the study of the finite-energy spectral and dynamical properties. The pseudofermion S matrix is expressed as a commutative product of S matrices, each corresponding to an elementary two-pseudofermion scattering event. This commutative factorization is stronger than the usual factorization associated with Yang-Baxter Equation for the original spin 1/2 electron bare S matrix. Our results reveal the scattering mechanisms which control the exotic finite-energy spectral properties of the low-dimensional complex materials and correlated systems of cold fermionic atoms on an optical lattice. Importantly, the exotic scatterers and scattering centers predicted by the theory were observed by angle-resolved photoelectron spectroscopy in low-dimensional organic metals.

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I. INTRODUCTION

The one-dimensional (1D) Hubbard model is solvable by coordinate Bethe ansatz (BA) [1, 2]. The solution of the model can also be achieved by the inverse-scattering algebraic BA [3]. In terms of electronic scattering the model describes a very complex non-perturbative strongly correlated problem. The main goal of this paper is to show that in terms of the pseudofermions associated with the holons and spinons introduced in Refs. [4, 5, 6] the scattering problem considerably simplifies and involves two-pseudofermion zero-momentum forward scattering only. In the last twenty years the low-energy behavior of the model correlation functions has been the subject of many studies [7, 8, 9, 10, 11, 12, 13]. Recently, the use of the pseudofermion representation extended such studies to finite energy [14, 15, 16].

The generalization of the electron - rotated-electron unitary transformation introduced in Ref. [17] for all values of the on-site repulsion U plays an important role in the construction of the holon and spinon representation of Ref. [4]. Such a representation is faithful for the whole Hilbert space, including the subspace generated by application onto the BA-solution states of the off-diagonal generators of the spin and η -spin $SU(2)$ algebras [18, 19]. In turn, there is one pseudofermion representation for each initial ground state whose subspace is spanned by the excited energy eigenstates contained in the excitations generated by application of one- and two-electron operators onto the former state [6, 15]. The pseudofermion dynamical theory (PDT) [14, 15, 16] is a suitable starting point for the study of the dynamical and spectral properties of the model for all values of the momentum, energy, on-site repulsion U , and electronic density n . However, the relation of the PDT to the scattering mechanisms remains an open question. The theory is a generalization for all values of the on-site repulsion U of the $U/t \gg 1$ method of Ref. [20]. Here t is the first-neighbor transfer integral.

The finite-energy PDT reproduces the well-known behavior of spectral and correlation functions in the limit of low energy [16], which was previously obtained [7, 8, 9, 10, 11, 12, 13] by use of methods such as conformal-field theory [21] and bosonization [22]. The theory was successfully applied to the description of the unusual finite-energy

spectral properties of low-dimensional complex materials [23, 24, 25]: For the one-electron removal spectral function the singular spectral features predicted by the PDT show quantitative agreement for the whole energy band width with the peak dispersions observed by angle-resolved photoelectron spectroscopy in the quasi-1D organic conductor TTF-TCNQ [23, 24]. (Results for the TTF-TCNQ spectrum consistent with those of the PDT were obtained by the dynamical density matrix renormalization group method [26].) Moreover, the theory was also used in the description of the phase diagram of other low-dimensional complex materials [25] and is of interest for the study of the spectral properties of the new quantum systems described by cold fermionic atoms on an optical lattice [27]. (New experiments involving cold fermionic atoms [such as ^6Li] on an optical lattice formed by interfering laser fields are in progress [28].)

We are able to calculate explicitly the pseudofermion and pseudofermion hole S matrices and overall phase shifts and to clarify how these quantities control the finite-energy spectral properties. Interestingly, the unusual independent charge and spin spectral features observed by angle-resolved photoelectron spectroscopy in low-dimensional organic metals [23, 24] correspond to the exotic independent charge and spin pseudofermion scatters and scattering centers introduced here. This paper contains a detailed presentation of the preliminary results presented in short form elsewhere [29].

The relation of the holon, spinon, and pseudofermion description of Refs. [4, 5, 6, 14, 15, 16] used here to the conventional holon and spinon representation and scattering theory of Refs. [30, 31, 32] was very recently clarified in Ref. [33]. Such an investigation confirms that both representations are faithful and thus that there is no inconsistency between the two corresponding definitions of quantum objects. Moreover, that study confirms that the holon, spinon, and pseudofermion description of Refs. [4, 5, 6, 14, 15, 16] is the most suitable for the study of the finite-energy spectral and dynamical properties.

The paper is organized as follows: In Sec. II we introduce the 1D Hubbard model and summarize the basic information about the pseudofermion description needed for our studies. The general pseudofermion scattering theory and the pseudofermion S matrix are introduced in Sec. III. In that section we also discuss the relation between the pseudofermion scattering theory and the spectral and dynamical properties. In Sec. IV we introduce and study the pseudofermion phase shifts. Finally, Sec. V contains the concluding remarks.

II. THE 1D HUBBARD MODEL AND THE PSEUDOFERMION DESCRIPTION

In this section we introduce the 1D Hubbard model and summarize the concepts and results concerning rotated electrons [4, 17] and the pseudofermion description [6] that are needed for our studies.

A. THE 1D HUBBARD MODEL AND ROTATED ELECTRONS

The exotic quantum objects associated with the BA solution of the 1D Hubbard model are related to the electrons through the rotated electrons [4, 5, 6]. Let us start by introducing the model.

1. THE 1D HUBBARD MODEL

In a chemical potential μ and magnetic field H the 1D Hubbard Hamiltonian can be written as,

$$\hat{H} = \hat{H}_{SO(4)} + \sum_{\alpha=c, s} \mu_{\alpha} \hat{S}_{\alpha}^z, \quad (1)$$

where

$$\hat{H}_{SO(4)} = \hat{H}_H - \frac{U}{2} \left[\hat{N} - \frac{N_a}{2} \right]; \quad \hat{H}_H = \hat{T} + U \hat{D}. \quad (2)$$

Here \hat{H}_H is the “simple” Hubbard model, $\hat{T} = -t \sum_{\sigma=\uparrow, \downarrow} \sum_{j=1}^{N_a} \left[c_{j, \sigma}^{\dagger} c_{j+1, \sigma} + h.c. \right]$ is the *kinetic-energy* operator, $\hat{D} = \sum_{j=1}^{N_a} c_{j, \uparrow}^{\dagger} c_{j, \uparrow} c_{j, \downarrow}^{\dagger} c_{j, \downarrow} = \sum_{j=1}^{N_a} \hat{n}_{j, \uparrow} \hat{n}_{j, \downarrow}$ is the electron double-occupation operator, and the operator $\hat{n}_{j, \sigma} = c_{j, \sigma}^{\dagger} c_{j, \sigma}$ counts the number of spin-projection σ electrons at lattice site j . The operator $c_{j, \sigma}^{\dagger}$ (and $c_{j, \sigma}$) that appears in the above equations creates (and annihilates) a spin-projection σ electron at lattice site $j = 1, 2, \dots, N_a$. We consider that the number of lattice sites N_a is large and even. On the right-hand side of Eq. (1), $\mu_c = 2\mu$, $\mu_s = 2\mu_0 H$, μ_0 is the Bohr magneton, and the number operators $\hat{S}_c^z = -\frac{1}{2}[N_a - \hat{N}]$ and $\hat{S}_s^z = -\frac{1}{2}[\hat{N}_{\uparrow} - \hat{N}_{\downarrow}]$ are the diagonal generators of

the η -spin and spin $SU(2)$ algebras [18, 19], respectively. Here the electronic number operators read $\hat{N} = \sum_{\sigma=\uparrow,\downarrow} \hat{N}_\sigma$ and $\hat{N}_\sigma = \sum_{j=1}^{N_a} \hat{n}_{j,\sigma}$. The momentum operator is given by $\hat{P} = \sum_{\sigma=\uparrow,\downarrow} \sum_k \hat{n}_\sigma(k) k$, where the spin-projection σ momentum distribution operator reads $\hat{n}_\sigma(k) = c_{k,\sigma}^\dagger c_{k,\sigma}$ and the operator $c_{k,\sigma}^\dagger$ (and $c_{k,\sigma}$) creates (and annihilates) a spin-projection σ electron of momentum k .

Throughout this paper we use units of both Planck constant \hbar and lattice constant a one. We denote the electronic charge by $-e$, the lattice length by $L = N_a a = N_a$, and the η -spin value η (and spin value S) and η -spin projection η_z (and spin projection S_z) of the energy eigenstates by S_c and S_c^z (and S_s and S_s^z), respectively. The Hamiltonian $\hat{H}_{SO(4)}$ given in Eq. (2) commutes with the six generators of the η -spin and spin $SU(2)$ algebras and has $SO(4)$ symmetry [18, 19]. While the expressions of the two corresponding diagonal generators were given above, the off-diagonal generators of these two $SU(2)$ algebras read $\hat{S}_c^\dagger = \sum_j (-1)^j c_{j,\downarrow}^\dagger c_{j,\uparrow}$ and $\hat{S}_c = \sum_j (-1)^j c_{j,\uparrow} c_{j,\downarrow}$ for η spin and $\hat{S}_s^\dagger = \sum_j c_{j,\downarrow}^\dagger c_{j,\uparrow}$ and $\hat{S}_s = \sum_j c_{j,\uparrow} c_{j,\downarrow}$ for spin. The BA solvability of the 1D Hubbard model (1) is restricted to the Hilbert subspace spanned by the lowest-weight states (LWSs) [1, 2] or highest-weight states (HWSs) [3] of the η -spin and spin algebras, that is by the states whose S_α and S_α^z numbers are such that $S_\alpha = -S_\alpha^z$ or $S_\alpha = S_\alpha^z$, respectively, where $\alpha = c$ for η -spin and $\alpha = s$ for spin. Such states have electronic densities $n = N/L$ and spin densities $m = [N_\uparrow - N_\downarrow]/L$ in the domains $0 \leq n \leq 1$ and $0 \leq m \leq n$, respectively. The description of the states corresponding to the extended domains $0 \leq n \leq 1$; $1 \leq n \leq 2$ and $-n \leq m \leq n$; $-(2-n) \leq m \leq (2-n)$, respectively, is achieved by application onto the latter states of off-diagonal generators of the η -spin and spin $SU(2)$ algebras [4]. The scattering processes studied in this paper result from ground-state - excited-state transitions. For simplicity, here we consider initial ground states with densities in the domains $0 \leq n \leq 1$ and $0 \leq m \leq n$, respectively. (Some of our results correspond to initial ground states with densities in the ranges $0 < n < 1$ and $0 < m < n$.)

2. ROTATED ELECTRONS

Each lattice site $j = 1, 2, \dots, N_a$ of the model (1) can either be doubly occupied, unoccupied, or singly occupied by a spin-down or spin-up electron. The maximum number of electrons is $2N_a$ and corresponds to density $n = 2$. Besides the N electrons, it is useful to consider $N^h = [2N_a - N]$ *electronic holes*. (Here we use the designation *electronic hole* instead of *hole*, in order to distinguish this type of hole from the pseudofermion hole.) Our definition of electronic hole is such that when a lattice site is unoccupied, we say that it is occupied by two electronic holes. If a lattice site is singly occupied, we say that it is occupied by an electron and an electronic hole. If a lattice site is doubly occupied, it is unoccupied by electronic holes. The same definition applies to the rotated-electronic holes.

The electron - rotated-electron unitary transformation maps the electrons onto rotated electrons such that rotated-electron double occupation, non-occupation, and spin-up and spin-down single occupation are good quantum numbers for all values of U/t [4, 5]. The lattice occupied by rotated electrons is identical to the original electronic lattice. We call $c_{j,\sigma}^\dagger$ the electrons that occur in the 1D Hubbard model (1) and (2), while the operator $\tilde{c}_{j,\sigma}^\dagger$ such that $\tilde{c}_{j,\sigma}^\dagger = \hat{V}^\dagger(U/t) c_{j,\sigma}^\dagger \hat{V}(U/t)$ represents the rotated electrons, where $\hat{V}(U/t)$ denotes the electron - rotated-electron unitary operator. Similarly, $c_{j,\sigma}^\dagger = \hat{V}(U/t) \tilde{c}_{j,\sigma}^\dagger \hat{V}^\dagger(U/t)$. Note that for $m = 0$ $c_{j,\sigma}^\dagger$ and $\tilde{c}_{j,\sigma}^\dagger$ are only identical in the $U/t \rightarrow \infty$ limit where electron double occupation becomes a good quantum number. The operators $\hat{V}^\dagger(U/t)$ and $\hat{V}(U/t)$ are uniquely defined for all values of U/t by Eqs. (21)-(23) of Ref. [4]. The electron - rotated-electron unitary transformation was introduced in Ref. [17]. The rotated-electron double occupation operator \hat{D} given in Eq. (20) of Ref. [4] commutes with the 1D Hubbard model. We denote the rotated-electron double occupation by D_r .

B. THE PSEUDOFERMION DESCRIPTION

Here we summarize the pseudofermion properties that are needed for the studies of this paper. The pseudoparticles studied in Refs. [4, 5] and the pseudofermions used in the investigations of Refs. [15, 16] are closely related. While the pseudoparticles have discrete bare-momentum values q_j such that $q_{j+1} - q_j = 2\pi/L$, the corresponding pseudofermions have *canonical-momentum* values $\bar{q}_j = q_j + Q_{\alpha\nu}^\Phi(q_j)/L$. (The designation "bare-momentum" follows from the discrete values q_j in units of $2\pi/L$ corresponding to quantum numbers, as given in Eq. (B.1) of Ref. [4]; The designation "canonical-momentum" stems from the analogy of the momentum shift $Q_{\alpha\nu}^\Phi(q_j)/L$ with the shift within the canonical momentum of electrons in the presence of a vector potential.) Here $Q_{\alpha\nu}^\Phi(q_j)/L$ is the functional given in Eq. (14) of Ref. [6] and $\alpha\nu$ labels the pseudofermion branch, as discussed below. Although that functional is of the order $1/L$, the discrete canonical-momentum are such that $\bar{q}_{j+1} - \bar{q}_j = 2\pi/L + O(1/L^2)$. Except for the slightly different discrete canonical-momentum values \bar{q}_j and discrete bare-momentum values q_j , the pseudofermions have the same properties

as the corresponding pseudoparticles. For instance, they have the same values of charge, η -spin, or spin and for the branches other than the $c0$ branch, also the same holon or spinon contents. The pseudofermion description refers to a Hilbert subspace called in Ref. [6] *pseudofermion subspace* (PS). (All one-, two-, and any other finite-number-electron excitations are contained in the PS.) In the PS the energy eigenstates are described by the same pseudoparticle [4] and pseudofermion [6] occupancy configurations and the $\alpha\nu$ pseudoparticles and $\alpha\nu$ pseudofermions are related by a unitary transformation [6]. Thus, the basic pseudofermion properties summarized below have many similarities with the corresponding pseudoparticle properties studied in Refs. [4, 5].

1. $c0$ PSEUDOFERMIONS, COMPOSITE PSEUDOFERMIONS, YANG HOLONS, AND HL SPINONS

A key result needed for our study is that the energy eigenstates that span the PS can be described in terms of occupancy configurations of holons, spinons, and $c0$ pseudofermions [4, 6]. We recall that the holons and spinons considered here are different from those of the conventional spinon-holon representation used in the studies of Refs. [30, 31, 32] and that the relation between the two alternative holon and spinon representations is clarified in Ref. [33]. For the simplest excited energy eigenstates, the holon (and spinon) of the conventional representation involves mixing of the $c0$ pseudofermion hole and Yang holon (and $s1$ pseudofermion hole and HL spinon) considered below. The holons (and spinons) introduced in Ref. [4] have η -spin $1/2$, η -spin projection $\pm 1/2$, charge $\pm 2e$, and spin zero (spin $1/2$, spin projection $\pm 1/2$, and no charge degrees of freedom). We use the notation $\pm 1/2$ holons (and $\pm 1/2$ spinons) according to the value of η -spin projection (and spin projection). The rotated-electron double occupation D_r equals the number of $-1/2$ holons. Within the description of charge transport in terms of electrons (and electronic holes), the $c0$ pseudofermions carry charge $-e$ (and $+e$) and have no spin or η -spin degrees of freedom. Moreover, the $c\nu$ pseudofermions (and $s\nu$ pseudofermions) are η -spin zero (and spin zero) composite objects of an equal number $\nu = 1, 2, \dots$ of $-1/2$ holons and $+1/2$ holons (and $-1/2$ spinons and $+1/2$ spinons). Within the description of charge transport in terms of electrons (and electronic holes), the $c\nu$ pseudofermions carry charge $-2\nu e$ (and $+2\nu e$) where $\nu = 1, 2, \dots$. In this paper we use the notation $\alpha\nu$ pseudofermion, where $\alpha = c, s$ and $\nu = 0, 1, 2, \dots$ for the $c\nu$ branches and $\nu = 1, 2, \dots$ for the $s\nu$ branches. The $\pm 1/2$ holons (and $\pm 1/2$ spinons) which are not part of 2ν -holon composite $c\nu$ pseudofermions (and 2ν -spinon composite $s\nu$ pseudofermions) are called $\pm 1/2$ Yang holons (and $\pm 1/2$ HL spinons). In the designations *HL spinon* and *Yang holon*, HL stands for Heilmann and Lieb and Yang refers to C. N. Yang, respectively, who are the authors of Refs. [18, 19]. We denote the number of $\alpha\nu$ pseudofermions by $N_{\alpha\nu}$ and the number $\pm 1/2$ Yang holons ($\alpha = c$) and $\pm 1/2$ HL spinons ($\alpha = s$) by $L_{\alpha, \pm 1/2}$. Note that N_{c0} equals the number of rotated-electron singly occupied sites, $[N_a - N_{c0}]$ equals the number of rotated-electron doubly occupied plus unoccupied sites, and $L_{\alpha, \pm 1/2} = S_{\alpha} \mp S_{\alpha}^z$. We call $M_{\alpha, \pm 1/2}$ the number of $\pm 1/2$ holons ($\alpha = c$) and $\pm 1/2$ spinons ($\alpha = s$) such that $M_{\alpha, \pm 1/2} = L_{\alpha, \pm 1/2} + \sum_{\nu=1}^{\infty} \nu N_{\alpha\nu}$. These numbers are given by $M_{c, -1/2} = [N - N_{c0}]/2$, $M_{c, +1/2} = [N^h - N_{c0}]/2$, $M_{s, -1/2} = [N_{c0} - N_{\uparrow} + N_{\downarrow}]/2$, and $M_{s, +1/2} = [N_{c0} + N_{\uparrow} - N_{\downarrow}]/2$. Furthermore, $M_{\alpha} = [M_{\alpha, -1/2} + M_{\alpha, +1/2}]$ denotes the number of holons ($\alpha = c$) or spinons ($\alpha = s$) such that $M_c = [N_a - N_{c0}]$ and $M_s = N_{c0}$ and $L_{\alpha} = [L_{\alpha, -1/2} + L_{\alpha, +1/2}]$ denotes the number of Yang holons ($\alpha = c$) or HL spinons ($\alpha = s$) such that $L_c = 2S_c = 2\eta$ and $L_s = 2S_s = 2S$. An important point is that for the ground state and densities such that $0 \leq n \leq 1$ and $0 \leq m \leq n$ one finds that $N_{c0} = N$, $N_{s1} = N_{\downarrow}$, $M_{c, +1/2} = L_{c, +1/2} = [N_a - N]$, $M_{s, -1/2} = N_{\downarrow}$, $M_{s, +1/2} = N_{\uparrow}$, $L_{s, +1/2} = [N_{\uparrow} - N_{\downarrow}]$, and $N_{\alpha\nu} = M_{c, -1/2} = L_{\alpha, -1/2} = 0$ for $\alpha\nu \neq c0, s1$ and $\alpha = c, s$.

Often in this paper we use the notation $\alpha\nu \neq c0, s1$ branches, which refers to all $\alpha\nu$ branches except the $c0$ and $s1$ branches. Moreover, the summations (and products) $\sum_{\alpha\nu}$, $\sum_{\alpha\nu=c0, s1}$, and $\sum_{\alpha\nu \neq c0, s1}$ (and $\prod_{\alpha\nu}$, $\prod_{\alpha\nu=c0, s1}$, and $\prod_{\alpha\nu \neq c0, s1}$) run over all $\alpha\nu$ branches with finite $\alpha\nu$ pseudofermion occupancy in the corresponding state or subspace, the $c0$ and $s1$ branches only, and all $\alpha\nu$ branches with finite $\alpha\nu$ pseudofermion occupancy in the corresponding state or subspace except the $c0$ and $s1$ branches, respectively.

2. THE PSEUDOFERMION CANONICAL MOMENTUM AND ASSOCIATED FUNCTIONALS

As mentioned above, the $\alpha\nu$ pseudofermion discrete canonical-momentum values \bar{q}_j are of the following form,

$$\bar{q}_j = \bar{q}(q_j) = q_j + \frac{Q_{\alpha\nu}^{\Phi}(q_j)}{L} = \frac{2\pi}{L} I_j^{\alpha\nu} + \frac{Q_{\alpha\nu}^{\Phi}(q_j)}{L}; \quad j = 1, 2, \dots, N_{\alpha\nu}^*. \quad (3)$$

where $I_j^{\alpha\nu}$ are integers or half-odd integers [4], $N_{\alpha\nu}^* = N_{\alpha\nu} + N_{\alpha\nu}^h$, and $N_{\alpha\nu}^h$ denotes the number of $\alpha\nu$ pseudofermion holes. The latter number equals the corresponding number of $\alpha\nu$ pseudoparticle holes given in Eqs. (B7) and (B8) of Ref. [4]. Note that besides equaling the number of discrete canonical-momentum values in the $\alpha\nu$ canonical-momentum band, $N_{\alpha\nu}^* = N_{\alpha\nu} + N_{\alpha\nu}^h$ also equals the number of sites of the $\alpha\nu$ effective lattice [6], which plays an

important role in the pseudofermion description. In addition to the $\alpha\nu$ pseudofermions of canonical momentum \bar{q} , there are local $\alpha\nu$ pseudofermions, whose creation and annihilation operators correspond to the sites of the effective $\alpha\nu$ lattice. Such a lattice has spatial coordinates $x_j = a_{\alpha\nu} j$ where $j = 1, 2, \dots, N_{\alpha\nu}^*$ and $N_{\alpha\nu}^*$ is the number of sites defined in Eqs. (B.6)-(B.8) and (B.11) of Ref. [4] and $a_{\alpha\nu} = L/N_{\alpha\nu}^*$ is the effective $\alpha\nu$ lattice constant. Each $\alpha\nu$ pseudofermion band is associated with an effective $\alpha\nu$ lattice whose length $L = N_{\alpha\nu}^* a_{\alpha\nu}$ is the same as that of the original real-space lattice. [6]. The relation between the momentum and local pseudofermion operators is given in Eq. (34) of Ref. [6].

The discrete bare-momentum q_j is a good quantum number whose allowed occupancies are one and zero only. (Also the corresponding discrete canonical-momentum \bar{q}_j has allowed occupancies one and zero only.) Thus, for the bare-momentum occupancy configuration describing a given energy eigenstate the bare-momentum distribution function $N_{\alpha\nu}(q_j)$ is such that $N_{\alpha\nu}(q_j) = 1$ for occupied bare-momentum values and $N_{\alpha\nu}(q_j) = 0$ for unoccupied bare-momentum values. We denote the ground-state bare-momentum distribution function by $N_{\alpha\nu}^0(q_j)$. It is given in Eqs. (C.1)-(C.3) of Ref. [4]. Although the $\alpha\nu$ pseudoparticles carry bare-momentum q_j , one can also label the corresponding $\alpha\nu$ pseudofermions by such a bare-momentum. This is because there is a one-to-one correspondence between the bare momentum q_j and the pseudofermion canonical momentum $\bar{q}_j = q_j + Q_{\alpha\nu}^\Phi(q_j)/L$. Thus, when one refers to the pseudofermion bare-momentum q_j , one means that q_j is the bare-momentum value that corresponds to the pseudofermion canonical momentum $\bar{q}_j = q_j + Q_{\alpha\nu}^\Phi(q_j)/L$. The pseudofermion canonical-momentum shift functional $Q_{\alpha\nu}^\Phi(q_j)/L$ is given by,

$$\frac{Q_{\alpha\nu}^\Phi(q_j)}{L} = \frac{2\pi}{L} \sum_{\alpha'\nu'}^{N_{\alpha'\nu'}^*} \sum_{j'=1} \Phi_{\alpha\nu, \alpha'\nu'}(q_j, q_{j'}) \Delta N_{\alpha'\nu'}(q_{j'}), \quad (4)$$

where,

$$\Delta N_{\alpha\nu}(q_j) \equiv N_{\alpha\nu}(q_j) - N_{\alpha\nu}^0(q_j), \quad (5)$$

is the $\alpha\nu$ bare-momentum distribution-function deviation. A PS excited energy eigenstate is uniquely defined by the values of the set of deviations $\{\Delta N_{\alpha\nu}(q_j)\}$ for all values of q_j corresponding to the $\alpha\nu$ branches with finite pseudofermion occupancy in the state and by the values $L_{c, -1/2}$ and $L_{s, -1/2}$. Moreover, the quantity $\Phi_{\alpha\nu, \alpha'\nu'}(q, q')$ on the right-hand side of Eq. (4) is a function of both the bare-momentum values q and q' given by,

$$\Phi_{\alpha\nu, \alpha'\nu'}(q, q') = \bar{\Phi}_{\alpha\nu, \alpha'\nu'} \left(\frac{4t \Lambda_{\alpha\nu}^0(q)}{U}, \frac{4t \Lambda_{\alpha'\nu'}^0(q')}{U} \right), \quad (6)$$

where the function $\bar{\Phi}_{\alpha\nu, \alpha'\nu'}(r, r')$ is the unique solution of the integral equations (A1)-(A13) of Ref. [6]. The ground-state rapidity functions $\Lambda_{\alpha\nu}^0(q)$ appearing in Eq. (6), where $\Lambda_{c0}^0(q) \equiv \sin k^0(q)$ for $\alpha\nu = c0$, are defined in terms of the inverse functions of $k^0(q)$ and $\Lambda_{\alpha\nu}^0(q)$ for $\nu > 0$ in Eqs. (A.1) and (A.2) of Ref. [15].

It is found below that $\pi \Phi_{\alpha\nu, \alpha'\nu'}(q, q')$ [or $-\pi \Phi_{\alpha\nu, \alpha'\nu'}(q, q')$] is an elementary *two-pseudofermion phase shift* such that q is the bare-momentum value of a $\alpha\nu$ pseudofermion or $\alpha\nu$ pseudofermion hole scattered by a $\alpha'\nu'$ pseudofermion [or $\alpha'\nu'$ pseudofermion hole] of bare-momentum q' created under a ground-state - excited-energy-eigenstate transition. As discussed in Sec. IV-D, there are no $c\nu \neq c0$ (and $s\nu \neq s1$) bare-momentum bands for $n = 1$ (and $m = 0$) ground states. Indeed, $N_{c\nu}^* = 0$ (and $N_{s\nu}^* = 0$) for such states and then the corresponding ground-state rapidity functions $\Lambda_{c\nu}^0(q)$ (and $\Lambda_{s\nu}^0(q)$) cannot be defined. Fortunately, expression (6) remains valid in that case provided that the ground-state rapidity functions are suitably replaced by those of the excited states. (Indeed, we find below that the functions (6) are phase shifts originated by well-defined ground-state - excited-state transitions; Thus, in the particular case of the $n = 1$ and/or $m = 0$ ground states the quantities (6) are functionals rather than functions, with the rapidity functions for the $c\nu \neq c0$ and/or $s\nu \neq s1$ branches being those of the excited state under consideration.)

The form of the functional (4) reveals that for the initial ground state the discrete canonical-momentum value \bar{q}_j and corresponding discrete bare-momentum value q_j are such that $\bar{q}_j = q_j$. The ground-state continuum bare-momentum and canonical-momentum values belong to the domain $q \in [-q_{\alpha\nu}^0, +q_{\alpha\nu}^0]$, where the limiting value $q_{\alpha\nu}^0$ reads,

$$q_{c0}^0 = \pi; \quad q_{s1}^0 = k_{F\uparrow}; \quad q_{c\nu}^0 = [\pi - 2k_F], \quad \nu > 0; \quad q_{s\nu}^0 = [k_{F\uparrow} - k_{F\downarrow}], \quad \nu > 1. \quad (7)$$

The ground-state is described by a compact $c0$ and $s1$ pseudofermion finite occupancy for $q \in [-q_{F\alpha\nu}^0, +q_{F\alpha\nu}^0]$, while the remaining branches have vanishing ground-state occupancy. Here the $c0$ and $s1$ *Fermi* points are given by,

$$q_{F c0}^0 = 2k_F; \quad q_{F s1}^0 = k_{F\downarrow}. \quad (8)$$

Both the limiting values of Eq. (7) and the ground-state *Fermi* values of Eq. (8) are given except for corrections of order $1/L$. The limiting bare-momentum values and ground-state *Fermi* bare-momentum including the $1/L$ corrections are provided in Eqs. (B.14)-(B.17) and (C.4)-(C.11), respectively, of Ref. [4].

3. THE PSEUDOFERMION ENERGY SPECTRUM

The PS general energy spectrum can be expressed in terms of the pseudofermion canonical-momentum distribution-function deviations as follows [6],

$$\begin{aligned} \Delta E = & \sum_{\bar{q}=-\pi}^{+\pi} \epsilon_{c0}(\bar{q}) \Delta \mathcal{N}_{c0}(\bar{q}) + \sum_{\bar{q}=-k_{F\uparrow}}^{+k_{F\uparrow}} \epsilon_{s1}(\bar{q}) \Delta \mathcal{N}_{s1}(\bar{q}) \\ & + E_h + \sum_{\nu=2}^{\infty} \sum_{\bar{q}=-[k_{F\uparrow}-k_{F\downarrow}]}^{+[k_{F\uparrow}-k_{F\downarrow}]} \epsilon_{s\nu}^0(\bar{q}) \Delta \mathcal{N}_{s\nu}(\bar{q}) + E_{uhb} + \sum_{\nu=1}^{\infty} \sum_{\bar{q}=-[\pi-2k_F]}^{+[\pi-2k_F]} \epsilon_{c\nu}^0(\bar{q}) \Delta \mathcal{N}_{c\nu}(\bar{q}). \end{aligned} \quad (9)$$

Here $\Delta \mathcal{N}_{\alpha\nu}(\bar{q}) = \Delta N_{\alpha\nu}(q)$ and the $\alpha\nu$ energy bands are defined in Eqs. (C.15)-(C.21) of Ref. [4] and plotted in Figs. 6 to 9 of Ref. [5] for $m = 0$. The zero-energy level of these energy bands is such that,

$$\epsilon_{c0}(\pm 2k_F) = \epsilon_{s1}(\pm k_{F\downarrow}) = \epsilon_{c\nu}^0(\pm[\pi - 2k_F]) = \epsilon_{s\nu}^0(\pm[k_{F\uparrow} - k_{F\downarrow}]) = 0. \quad (10)$$

The rotated-electron double occupation D_r and the number S_r of spin-down rotated-electron singly occupied sites whose rotated electrons are not associated with the $s1$ pseudofermions play an important role in the finite-energy physics and are given by,

$$D_r \equiv M_{c,-1/2} = [L_{c,-1/2} + \sum_{\nu=1}^{\infty} \nu N_{c\nu}]; \quad S_r \equiv [M_{s,-1/2} - N_{s1}] = [L_{s,-1/2} + \sum_{\nu=2}^{\infty} \nu N_{s\nu}]. \quad (11)$$

These quantities fully determine the value for the upper-Hubbard band (UHB) energy gap E_{uhb} and spin gap E_h of the PS energy spectrum (9) as follows,

$$E_{uhb} = 2\mu D_r; \quad E_h = 2\mu_0 H S_r. \quad (12)$$

Note that $E_{uhb} = E_h = 0$ for the initial ground state. In equation (11) $L_{\alpha,-1/2}$ are the numbers of $-1/2$ Yang holons ($\alpha = c$) and $-1/2$ HL spinons ($\alpha = s$) of the excited energy eigenstate, and $N_{\alpha\nu}$ is the number of $\alpha\nu$ pseudofermions of the same state for the $\alpha\nu \neq c0, s1$ branches.

III. THE PSEUDOFERMION SCATTERING THEORY: THE PSEUDOFERMION S MATRIX

In this section we introduce the pseudofermion scattering theory. We derive the pseudofermion and pseudofermion-hole S matrices and discuss their relation to the spectral properties. Our analysis of the problem follows the standard quantum non-relativistic scattering theory of spin-less particles [34]. A ground-state - excited-energy-eigenstate transition involves a set of elementary two-pseudofermion scattering events. Such a transition is divided below into three steps. The first and second steps have a scatter-less character and lead to the excitation momentum and energy associated with the transition. Through these two steps the ground-state - excited-energy-eigenstate transition reaches the many-pseudofermion “in” state, which contains the one-pseudofermion “in” asymptote states of the pseudofermion scattering theory. The third step corresponds to a well defined set of elementary two-pseudofermion scattering events which give rise to the “out” state and conserve both the momentum and the energy. The latter many-pseudofermion state is a excited energy eigenstate and contains the one-pseudofermion “out” asymptote states of the pseudofermion scattering theory. An important point for applications to the study of the finite-energy spectral and dynamical properties is that all “in” and “out” states of the theory are energy eigenstates

We start our analysis by a discussion of the PS subspaces and the pseudofermion creation and annihilation operator anti-commutation relations.

A. PS SUBSPACES AND THE PSEUDOFERMION OPERATOR ANTICOMMUTATORS

Several PS subspaces play an important role in the pseudofermion theory. An *electronic ensemble space* is a subspace spanned by all energy eigenstates with the same values for the electronic numbers N_{\uparrow} and N_{\downarrow} . A *CPHS ensemble space* is a subspace spanned by all energy eigenstates with the same values for the numbers $\{M_{\alpha,\pm 1/2}\}$ of $\pm 1/2$ holons ($\alpha = c$) and $\pm 1/2$ spinons ($\alpha = s$) [4, 5]. (In CPHS ensemble space, CPHS refers to $c0$ pseudofermion, holon, and

spinon.) A *CPHS ensemble subspace* is spanned by all energy eigenstates with the same values for the sets of numbers N_{c0} , $\{N_{c\nu}\}$, $\{N_{s\nu}\}$, $L_{c,-1/2}$, and $L_{s,-1/2}$ such that $\nu = 1, 2, \dots$

At zero absolute temperature the pseudofermion description corresponds to a ground-state normal-ordered theory [6]. Thus, there is a pseudofermion theory for each initial ground state. The minimum excitation energy value of the energy eigenstates that span a given CPHS ensemble subspace involves the gap parameters of Eq. (12) and is given by,

$$\omega_0 = \omega_0(D_r, S_r) = E_{uhb} + E_h = 2\mu D_r + 2\mu_0 H S_r. \quad (13)$$

For the ground state $D_r = S_r = 0$ and thus $\omega_0 = 0$. The application onto the latter state of an one-, two-, or any other finite-number-electron operator \mathcal{O}^\dagger generates an excitation which can be described as a suitable superposition of PS excited energy eigenstates. The PDT provides the matrix elements associated with the coefficients of such a superposition [14, 15, 16]. Therefore, the elementary ground-state - excited-energy-eigenstate transition plays a central role in the pseudofermion theory.

An excitation $\mathcal{O}^\dagger|GS\rangle$ associated with small values for the deviations ΔN_\uparrow and ΔN_\downarrow is contained in a well defined direct sum of CPHS ensemble subspaces,

$$\mathcal{S}_{cphs}^1 \oplus \mathcal{S}_{cphs}^2 \oplus \mathcal{S}_{cphs}^3 \oplus \mathcal{S}_{cphs}^4 \oplus \dots \quad (14)$$

Here \mathcal{S}_{cphs}^i with $i = 1, 2, 3, \dots$ corresponds to different CPHS ensemble subspaces. The pseudofermion, Yang holon, and HL spinon number deviations of all the CPHS ensemble subspaces of such a direct sum obey the sum rules (18) and (19) of Ref. [14] and the selection rules given in Eq. (21) of the same reference.

Let us consider a $\alpha\nu$ pseudofermion of canonical momentum \bar{q} and a $\alpha'\nu'$ pseudofermion of canonical momentum \bar{q}' such that the canonical-momentum values \bar{q} and \bar{q}' correspond to an excited energy eigenstate and the initial ground state, respectively. Following the results of Refs. [14, 15, 16], the anticommutators involving the creation and/or annihilation operators of these two pseudofermions play a key role in the study of the finite-energy spectral and dynamical properties. Such anticommutators read [6],

$$\{f_{\bar{q},\alpha\nu}^\dagger, f_{\bar{q}',\alpha'\nu'}\} = \frac{\delta_{\alpha\nu,\alpha'\nu'}}{N_{\alpha\nu}^*} e^{-i(\bar{q}-\bar{q}')/2} e^{iQ_{\alpha\nu}(q)/2} \frac{\sin(Q_{\alpha\nu}(q)/2)}{\sin([\bar{q}-\bar{q}']/2)}; \quad \{f_{\bar{q},\alpha\nu}^\dagger, f_{\bar{q}',\alpha'\nu'}^\dagger\} = \{f_{\bar{q},\alpha\nu}, f_{\bar{q}',\alpha'\nu'}\} = 0. \quad (15)$$

Here $Q_{\alpha\nu}(q)/2$ is the value of the following functional for the above excited energy eigenstate,

$$Q_{\alpha\nu}(q_j)/2 = Q_{\alpha\nu}^0/2 + Q_{\alpha\nu}^\Phi(q_j)/2, \quad (16)$$

where $Q_{\alpha\nu}^\Phi(q_j)/2$ is the functional given in Eq. (4) whose bare-momentum distribution function deviations correspond to that state and [6],

$$\begin{aligned} Q_{c0}^0 &= 0; & \sum_{\alpha=c,s} \sum_{\nu=1}^{\infty} \Delta N_{\alpha\nu} &\text{ even}; & Q_{c0}^0 &= \pm\pi; & \sum_{\alpha=c,s} \sum_{\nu=1}^{\infty} \Delta N_{\alpha\nu} &\text{ odd}; \\ Q_{\alpha\nu}^0 &= 0; & \Delta N_{c0} + \Delta N_{\alpha\nu} &\text{ even}; & Q_{\alpha\nu}^0 &= \pm\pi; & \Delta N_{c0} + \Delta N_{\alpha\nu} &\text{ odd}; & \alpha = c, s, \quad \nu > 0. \end{aligned} \quad (17)$$

When $Q_{\alpha\nu}^0 = \pm\pi$ for the $\alpha\nu \neq c0$ bands, the uniquely chosen and only permitted value $Q_{\alpha\nu}^0 = \pi$ or $Q_{\alpha\nu}^0 = -\pi$ is that which leads to symmetrical limiting discrete bare-momentum values $\pm[\pi/L][N_{\alpha\nu}^* - 1]$ for the excited-state bare-momentum band. (See Eq. (B.14) of Ref. [4].) In turn, for the $c0$ branch the bare-momentum band width is 2π . Thus, in this case $Q_{c0}^0 = \pi$ and $Q_{c0}^0 = -\pi$ lead to allowed occupancy configurations of alternative excited energy eigenstates. (In the particular case that the $c0$ band is full for the excited energy eigenstate, the two values $Q_{c0}^0 = \pi$ and $Q_{c0}^0 = -\pi$ refer to two equivalent representations of that state.)

The quantity $Q_{\alpha\nu}^0/L$ is the shift in the discrete bare-momentum value $q_j = [2\pi/L]I_j^{\alpha\nu}$ of Eq. (3) that arises due to the transition from the ground state to the excited energy eigenstate. Furthermore, $Q_{\alpha\nu}(q_j)/L$ is the corresponding shift in the discrete canonical-momentum values that occurs as a result of the same transition. It follows from Eq. (15) that the functional $Q_{\alpha\nu}(q)/2$ fully controls the pseudofermion anticommutators associated with the ground-state - excited-energy-eigenstate transition. The one- and two-electron matrix elements between the initial ground state and the excited energy eigenstates can be expressed in terms of the anticommutators (15) [14]. This justifies the importance of the functional $Q_{\alpha\nu}(q)/2$ given in Eq. (16), once it controls the quantum overlaps associated with the one- and two-electron finite-energy spectral properties [14, 15, 16].

Let us introduce the $\alpha\nu$ pseudofermion scattering theory. Within such a theory the functional $Q_{\alpha\nu}(q)/2$ is an overall $\alpha\nu$ pseudofermion or hole phase shift.

B. THE GROUND-STATE - VIRTUAL-STATE TRANSITION

From now on and until section IV-D our analysis refers to initial ground states with density values in the ranges $0 < n < 1$ and $0 < m < n$. The specific properties of the scattering theory for initial ground states corresponding to $n = 1$ and/or $m = 0$ are considered in that section. The preliminary analysis of the pseudofermion scattering problem presented in Ref. [29] divided each transition from the initial ground state to a PS excited energy eigenstate into two main steps. However, it is useful for our study to divide the first step considered in that reference into two processes. The first process considered here is a scatter-less finite-energy and finite-momentum excitation which transforms the ground state onto a well defined virtual state. For $\nu > 0$ branches, that excitation can involve a change in the number of discrete bare-momentum values given by,

$$\Delta N_{s1}^* = \Delta N_{c0} - \Delta N_{s1} - 2 \sum_{\nu=2}^{\infty} \Delta N_{s\nu}; \quad \Delta N_{\alpha\nu}^* = \Delta L_{\alpha} + 2 \sum_{\nu'=\nu+1}^{\infty} (\nu' - \nu) \Delta N_{\alpha\nu'}; \quad \alpha\nu \neq c0, s1. \quad (18)$$

For the initial ground state these numbers read,

$$N_{c\nu}^{0,*} = (N_a - N); \quad N_{s1}^{0,*} = N_{\uparrow}; \quad N_{s\nu}^{0,*} = (N_{\uparrow} - N_{\downarrow}), \quad \nu > 1, \quad (19)$$

and $N_{c0}^{0,*} = N_{c0}^*$ is given by $N_{c0}^* = N_a$ for the whole Hilbert space. Although the $\alpha\nu \neq c0, s1$ branches have no finite pseudofermion occupancy in the initial ground state, for densities $0 < n < 1$ and $0 < m < n$ one can define the values $N_{\alpha\nu}^* = N_{\alpha\nu}^h$ for the corresponding empty bands. For the $\alpha\nu \neq c0, s1$ branches, those are the numbers $N_{c\nu}^{0,*}$ and $N_{s\nu}^{0,*}$ given in Eq. (19). Thus, for $\alpha\nu \neq c0, s1$ branches with finite pseudofermion occupancy in the virtual state the deviations (18) and discrete bare-momentum shifts (17) are relative to the values of these empty bands.

In addition and following the change in the number of discrete bare-momentum values, this excitation also involves the pseudofermion creation and annihilation processes and pseudofermion particle-hole processes associated with PS excited states. The momentum and energy of this ground-state - virtual-state transition is given by,

$$\begin{aligned} \Delta P = & \sum_{q=-\pi}^{+\pi} q \Delta N_{c0}(q) + \sum_{q=-k_{F\uparrow}}^{+k_{F\uparrow}} q \Delta N_{s1}(q) + \sum_{\nu=2}^{\infty} \sum_{q=-[k_{F\uparrow}-k_{F\downarrow}]}^{+[k_{F\uparrow}-k_{F\downarrow}]} q \Delta N_{s\nu}(q) \\ & + \pi [L_{c,-1/2} + \sum_{\nu=1}^{\infty} \nu N_{c\nu}] + \sum_{\nu=1}^{\infty} \sum_{q=-[\pi-2k_F]}^{+[\pi-2k_F]} [\pi - q] \Delta N_{c\nu}(q), \end{aligned} \quad (20)$$

and

$$\begin{aligned} \Delta E = & \sum_{q=-\pi}^{+\pi} \epsilon_{c0}(q) \Delta N_{c0}(q) + \sum_{q=-k_{F\uparrow}}^{+k_{F\uparrow}} \epsilon_{s1}(q) \Delta N_{s1}(q) \\ & + E_h + \sum_{\nu=2}^{\infty} \sum_{q=-[k_{F\uparrow}-k_{F\downarrow}]}^{+[k_{F\uparrow}-k_{F\downarrow}]} \epsilon_{s\nu}^0(q) \Delta N_{s\nu}(q) + E_{uhb} + \sum_{\nu=1}^{\infty} \sum_{q=-[\pi-2k_F]}^{+[\pi-2k_F]} \epsilon_{c\nu}^0(q) \Delta N_{c\nu}(q), \end{aligned} \quad (21)$$

respectively, where all quantities were defined above. Except for $1/L$ energy corrections, the energy spectra (9) and (21) are identical. In this first scatter-less step the pseudofermions acquire the excitation momentum and energy needed for the second and third steps.

C. PSEUDOFERMION SCATTERING PROCESSES, S MATRICES, AND PHASE SHIFTS

In order to study the second and third processes of the ground-state - excited-energy-eigenstate transition, it is useful to express the many-pseudofermion states and operators in terms of one-pseudofermion states and operators, respectively. The PS energy and momentum eigenstates can be written as direct products of states spanned by the occupancy configurations of each of the $\alpha\nu$ branches with finite pseudofermion occupancy in the state under consideration. Moreover, the many-pseudofermion states spanned by occupancy configurations of each $\alpha\nu$ branch can be expressed as a direct product of $N_{\alpha\nu}^*$ one-pseudofermion states, each referring to one discrete bare-momentum value q_j , where $j = 1, 2, \dots, N_{\alpha\nu}^*$.

Within the pseudofermion description, the 1D Hubbard model in normal order relative to the initial ground state has no residual-interaction energy terms. Thus, when acting in the PS it has a uniquely defined expression of the general form [6],

$$: \hat{H} := \sum_{\alpha\nu} \sum_{j=1}^{N_{\alpha\nu}^*} \hat{H}_{\alpha\nu, q_j} + \sum_{\alpha} \hat{H}_{\alpha}, \quad (22)$$

where we denoted the ground-state normal ordered Hamiltonian by $: \hat{H} :$, $\hat{H}_{\alpha\nu, q_j}$ is the one-pseudofermion Hamiltonian which describes the $\alpha\nu$ pseudofermion or hole of bare-momentum q_j , and \hat{H}_{α} refers to the Yang holons ($\alpha = c$) and HL spinons ($\alpha = s$), which are scatter-less objects. Thus, for each many-pseudofermion PS virtual state reached in the first step of the transition from the ground state to the excited energy eigenstate, the number of Hamiltonians $\hat{H}_{\alpha\nu, q_j}$ equals that of one-pseudofermion states of the virtual state given by,

$$N_{c0}^* + N_{s1}^* + \sum_{\alpha\nu \neq c0, s1} \theta(|\Delta N_{\alpha\nu}|) N_{\alpha\nu}^*. \quad (23)$$

Here $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x = 0$ and $N_{c0}^* = N_{c0} + N_{c0}^h$, $N_{s1}^* = N_{s1} + N_{s1}^h$, and $N_{\alpha\nu}^* = N_{\alpha\nu} + N_{\alpha\nu}^h$ refer to the virtual state and corresponding excited energy eigenstate under consideration. The pseudofermion-hole numbers of these expressions read [4],

$$N_{c0}^h = N_a - N_{c0}; \quad N_{c\nu}^h = N_{c0}^h - \sum_{\nu'=1}^{\infty} (\nu + \nu' - |\nu - \nu'|) N_{c\nu'}; \quad N_{s\nu}^h = N_{c0} - \sum_{\nu'=1}^{\infty} (\nu + \nu' - |\nu - \nu'|) N_{s\nu'}. \quad (24)$$

For the pseudofermion description only momentum and energy contributions of order zero and one in $1/L$ are physical and thus our analysis refers to periodic boundary conditions and large values of L such that $L \gg 1$. The second scatter-less process generates the “in” state. Indeed, the one-pseudofermion states belonging to the many-pseudofermion “in” state are the “in” asymptote states of the pseudofermion scattering theory. The generator of the virtual-state - “in”-state transition is of the form,

$$\hat{S}^0 = \prod_{\alpha\nu} \prod_{j=1}^{N_{\alpha\nu}^*} \hat{S}_{\alpha\nu, q_j}^0, \quad (25)$$

where $\hat{S}_{\alpha\nu, q_j}^0$ is a well-defined one-pseudofermion unitary operator. Application of $\hat{S}_{\alpha\nu, q_j}^0$ onto the corresponding one-pseudofermion state of the many-pseudofermion virtual state shifts its discrete bare-momentum value q_j to the bare-momentum value $q_j + Q_{\alpha\nu}^0/L$, where $Q_{\alpha\nu}^0$ is given in Eq. (17).

Finally, the third step consists of a set of two-pseudofermion scattering events. It corresponds to the “in”-state - “out”-state transition, where the latter state is the PS excited energy eigenstate under consideration. The generator of that transition is the following operator,

$$\hat{S}^{\phi} = \prod_{\alpha\nu} \prod_{j=1}^{N_{\alpha\nu}^*} \hat{S}_{\alpha\nu, q_j}^{\phi}, \quad (26)$$

where $\hat{S}_{\alpha\nu, q_j}^{\phi}$ is a well-defined one-pseudofermion scattering unitary operator. The one-pseudofermion states belonging to the many-pseudofermion “out” state are the “out” asymptote pseudofermion scattering states. Application of $\hat{S}_{\alpha\nu, q_j}^{\phi}$ onto the corresponding one-pseudofermion state of the many-pseudofermion “in” state shifts its discrete bare-momentum value $q_j + Q_{\alpha\nu}^0/L$ to the “out”-state discrete canonical-momentum value $q_j + Q_{\alpha\nu}(q_j)/L$. It follows that the generator of the virtual-state - “out”-state transition is the unitary operator,

$$\hat{S} \equiv \hat{S}^{\phi} \hat{S}^0 = \prod_{\alpha\nu} \prod_{j=1}^{N_{\alpha\nu}^*} \hat{S}_{\alpha\nu, q_j}, \quad (27)$$

where $\hat{S}_{\alpha\nu, q_j}$ is the one-pseudofermion or hole unitary $\hat{S}_{\alpha\nu, q_j} = \hat{S}_{\alpha\nu, q_j}^{\phi} \hat{S}_{\alpha\nu, q_j}^0$ operator. Application of the latter operator onto the corresponding one-pseudofermion state of the many-pseudofermion virtual state shifts its discrete bare-momentum value q_j directly to the “out”-state discrete canonical-momentum value $q_j + Q_{\alpha\nu}(q_j)/L$.

The virtual state, “in” state, and “out” state are PS excited energy eigenstates, as further discussed below. Thus, that the one-pseudofermion states of the many-pseudofermion “in” state and “out” state are the “in” and “out” asymptote pseudofermion scattering states, respectively, implies that the one-pseudofermion Hamiltonian $\hat{H}_{\alpha\nu,q_j}$ plays the role of the unperturbed Hamiltonian \hat{H}_0 of the spin-less one-particle nonrelativistic scattering theory [34]. Indeed, the unitary $\hat{S}_{\alpha\nu,q_j}$ operator (and the scattering unitary $\hat{S}_{\alpha\nu,q_j}^\phi$ operator) commutes with the Hamiltonian $\hat{H}_{\alpha\nu,q_j}$ and thus the one-pseudofermion “in” and “out” asymptote scattering states are energy eigenstates of $\hat{H}_{\alpha\nu,q_j}$ and eigenstates of $\hat{S}_{\alpha\nu,q_j}$ (and $\hat{S}_{\alpha\nu,q_j}^\phi$). It follows that the matrix elements between one-pseudofermion states of $\hat{S}_{\alpha\nu,q_j}$ (and $\hat{S}_{\alpha\nu,q_j}^\phi$) are diagonal and thus these operators are fully defined by the set of their eigenvalues belonging to these states. The same applies to the generator \hat{S} (and \hat{S}^ϕ) given in Eq. (27) (and Eq. (26)). The matrix elements of that generator between many-pseudofermion virtual states (and “in” states) are also diagonal and thus it is fully defined by the set of its eigenvalues belonging to such states. Importantly, the virtual state and the “in” state of a given ground-state transition correspond to the same excited energy eigenstate of the 1D Hubbard model, which by construction is the “out” state. Indeed, the virtual state, “in” state, and “out” state only differ by mere overall phase factors whose general functional expression is given below. That the many-pseudofermion “in” and “out” states which are a direct product of one-pseudofermion “in” and “out” asymptote pseudofermion scattering states, respectively, are PS excited energy eigenstates plays a major role in the pseudofermion scattering theory.

Since $\hat{S}_{\alpha\nu,q_j}^\phi$ and $\hat{S}_{\alpha\nu,q_j}$ are unitary, each of their eigenvalues has modulus one and can be written as the exponent of a purely imaginary number given by,

$$\begin{aligned} S_{\alpha\nu}^\Phi(q_j) &= e^{iQ_{\alpha\nu}^\Phi(q_j)} = \prod_{\alpha'\nu'}^{N_{\alpha'\nu'}^*} \prod_{j'=1} S_{\alpha\nu,\alpha'\nu'}(q_j, q_{j'}) ; \quad j = 1, 2, \dots, N_{\alpha\nu}^* \\ S_{\alpha\nu}(q_j) &= e^{iQ_{\alpha\nu}(q_j)} = e^{iQ_{\alpha\nu}^0} S_{\alpha\nu}^\Phi(q_j) ; \quad j = 1, 2, \dots, N_{\alpha\nu}^* . \end{aligned} \quad (28)$$

Here $Q_{\alpha\nu}^\Phi(q_j)$ and $Q_{\alpha\nu}(q_j)$ are the functionals defined by Eqs. (4) and (16), respectively. By use of the former functional we find that,

$$S_{\alpha\nu,\alpha'\nu'}(q_j, q_{j'}) = e^{i2\pi \Phi_{\alpha\nu,\alpha'\nu'}(q_j, q_{j'}) \Delta N_{\alpha'\nu'}(q_{j'})} , \quad (29)$$

where the functions $\pi \Phi_{\alpha\nu,\alpha'\nu'}(q_j, q_{j'})$ are given in Eq. (6). The main point is that except for the occupancy configuration changes produced by the ground-state - virtual-state transition, the only effect of under a ground-state - excited-energy-eigenstate transition moving the $\alpha\nu$ pseudofermion or hole of initial virtual-state canonical-momentum $\bar{q}_j = q_j$ once around the length L lattice ring is that its wave function acquires the overall phase factor $S_{\alpha\nu}(q_j)$ given in Eq. (28). This property follows from the form of the energy spectrum of the pseudofermions, which in contrast to that of the corresponding pseudoparticles of Ref. [4] has no residual interaction terms [6].

The phase factor $S_{\alpha\nu,\alpha'\nu'}(q_j, q_{j'})$ of Eq. (29) in the wave function of the $\alpha\nu$ pseudofermion or hole of bare-momentum q_j results from an elementary two-pseudofermion zero-momentum forward-scattering event whose scattering center is a $\alpha'\nu'$ pseudofermion ($\Delta N_{\alpha'\nu'}(q_{j'}) = 1$) or $\alpha'\nu'$ pseudofermion hole ($\Delta N_{\alpha'\nu'}(q_{j'}) = -1$) created under the ground-state - excited-state transition. Thus, the third step of that transition involves a well-defined set of elementary two-pseudofermion scattering events where all $\alpha\nu$ pseudofermions and $\alpha\nu$ pseudofermion holes of bare-momentum $q_j + Q_{\alpha\nu}^0/L$ of the “in” state are the scatterers, which leads to the overall scattering phase factor $S_{\alpha\nu}^\Phi(q_j)$ in their wave function given in Eq. (28). That the scattering centers are the $\alpha'\nu'$ pseudofermions or pseudofermion holes of bare momentum $q_{j'} + Q_{\alpha\nu}^0/L$ created under the ground-state - “in”-state transition is confirmed by noting that $S_{\alpha\nu,\alpha'\nu'}(q_j, q_{j'}) = 1$ for $\Delta N_{\alpha'\nu'}(q_{j'}) = 0$. Thus, out of the scatterers whose number equals that of the one-pseudofermion states given in Eq. (23), the scattering centers are only those whose bare-momentum distribution-function deviation is finite. However, out of the above scatterers, only a subclass of scatterers contributes significantly to the spectral properties [14, 15].

The following properties play an important role in the pseudofermion scattering theory:

1. The elementary two-pseudofermion scattering processes associated with the phase factors (29) conserve the total energy and total momentum. Such an energy conservation is further discussed in Appendix A.
2. The elementary two-pseudofermion scattering processes are of zero-momentum forward-scattering type and thus conserve the individual “in” asymptote $\alpha\nu$ pseudofermion momentum value $q_j + Q_{\alpha\nu}^0/L$ and energy.
3. These processes also conserve the $\alpha\nu$ branch, usually called *channel* in the scattering language [34].
4. The scattering amplitude does not connect quantum objects with different η spin or spin.

5. For each $\alpha\nu$ pseudofermion or pseudofermion hole of virtual-state bare-momentum q_j , the S matrix associated with the ground-state - excited-energy-eigenstate transition is simply the phase factor $S_{\alpha\nu}(q_j)$ given in Eq. (28).

The one-particle phase factor $s_l(E)$ of Eq. (6.8) of Ref. [34] whose expression is given in Eq. (6.9) of the same reference corresponds to the one- $\alpha\nu$ -pseudofermion or hole phase factor $S_{\alpha\nu}^\Phi(q_j)$ with the energy E and the quantum numbers l and m replaced by the bare-momentum q_j . Indeed, while the $\alpha\nu$ pseudofermion or hole energy is uniquely defined by the absolute value $|q_j|$, in 1D the sign of q_j corresponds to the three-dimensional angular-momentum quantum numbers. Another difference is that $s_l(E)$ is associated with a single scattering event whereas $S_{\alpha\nu}^\Phi(q_j)$ results in general from several scattering events. Each of such events corresponds to a well defined factor $S_{\alpha\nu, \alpha'\nu'}(q_j, q_{j'})$ of form (29) in the expression of $S_{\alpha\nu}^\Phi(q_j)$ given in Eq. (28). There are as many of such factors as $\alpha'\nu'$ pseudofermion and hole scattering centers created under the transition to the virtual state and corresponding excited energy eigenstate under consideration. The factor 2 in the phase factor of Eq. (6.9) of Ref. [34] corresponds to the phase-shift definition of the standard nonrelativistic scattering theory for spin-less particles. As discussed below, we use here such a definition which introduces the overall scattering phase shift $\delta_{\alpha\nu}^\Phi(q_j) = Q_{\alpha\nu}^\Phi(q_j)/2$ and overall phase shift $\delta_{\alpha\nu}(q_j) = Q_{\alpha\nu}(q_j)/2$. However, if instead we insert a factor 1, we would introduce the overall scattering phase shift $Q_{\alpha\nu}^\Phi(q_j)$ and overall phase shift $Q_{\alpha\nu}(q_j)$ whose values are defined only to within addition of an arbitrary multiple of 2π . That is the choice for the phase shift definition used in Refs. [31, 32, 33]. For our definition the phase shifts are instead given only to within addition of an arbitrary multiple of π , as in Ref. [34].

Application of the unitary $\hat{S}_{\alpha\nu, q_j}$ operator onto its one-pseudofermion state of the many-pseudofermion virtual state generates the corresponding one-pseudofermion state of the many-pseudofermion “out” state. The latter one-pseudofermion state equals the former one multiplied by the phase factor $S_{\alpha\nu}(q_j)$ of Eq. (28). (Applying the scattering unitary $\hat{S}_{\alpha\nu, q_j}^\Phi$ operator onto its one-pseudofermion state of the many-pseudofermion “in” state also generates the corresponding one-pseudofermion state of the many-pseudofermion “out” state; The latter one-pseudofermion state equals the former one multiplied by the phase factor $S_{\alpha\nu}^\Phi(q_j)$ of Eq. (28).) It follows that the many-pseudofermion virtual states (and “in” states) are eigenstates of the generator \hat{S} (and \hat{S}^Φ) given in Eq. (27) (and Eq. (26)). The eigenvalue of \hat{S} belonging to a PS virtual state is given by,

$$S_T = e^{i2\delta_T} = \prod_{\alpha\nu} \prod_{j=1}^{N_{\alpha\nu}^*} S_{\alpha\nu}(q_j); \quad \delta_T = \sum_{\alpha\nu} \sum_{j=1}^{N_{\alpha\nu}^*} Q_{\alpha\nu}(q_j)/2. \quad (30)$$

Thus, the corresponding “out” state equals the virtual state multiplied by the phase factor S_T . Furthermore, the eigenvalue of \hat{S}^Φ belonging to a PS “in” state is given by,

$$S_T^\Phi = e^{i2\delta_T^\Phi} = \prod_{\alpha\nu} \prod_{j=1}^{N_{\alpha\nu}^*} S_{\alpha\nu}^\Phi(q_j); \quad \delta_T^\Phi = \sum_{\alpha\nu} \sum_{j=1}^{N_{\alpha\nu}^*} Q_{\alpha\nu}^\Phi(q_j)/2. \quad (31)$$

Again, the corresponding “out” state equals the “in” state multiplied by the phase factor S_T^Φ . Since the “out” state is by construction an energy eigenstate of the 1D Hubbard model, this result confirms that the corresponding virtual and “in” states are also energy eigenstates of the model: the “out” state only differs from the latter two states by the phase factors given in Eqs. (30) and (31), respectively. The general expressions (4) and (16) for the functionals $Q_{\alpha\nu}^\Phi(q_j)$ and $Q_{\alpha\nu}(q_j)$ define uniquely the eigenvalues S_T^Φ and S_T of \hat{S}^Φ and \hat{S} for the whole set of “in” states and virtual states, respectively, corresponding to the excited energy eigenstates that span the PS.

The factorization of the BA bare S matrix for the original spin 1/2 electrons is associated with the so called Yang-Baxter Equation (YBE) [30, 32]. On the other hand, the factorization of the $\alpha\nu$ pseudofermion or hole S matrix $S_{\alpha\nu}(q_j)$ given in Eq. (28), in terms of the elementary two-pseudofermion S matrices $S_{\alpha\nu, \alpha'\nu'}(q_j, q_{j'})$, Eq. (29), is commutative. Such commutativity is stronger than the symmetry associated with the YBE. The pseudofermion S matrix commutative factorization is required by the form of the pseudofermion occupancy configurations that describe the PS excited energy eigenstates. These states are direct products of the one-pseudofermion scattering states of the theory and are described by well defined occupancy configurations of rotated electrons. All such configurations have the same number of rotated-electron occupied sites, unoccupied sites, spin-up singly occupied sites, and spin-down singly occupied sites. However, the relative position of these quantum objects is different in each occupancy configuration. There is a one-to-one correspondence between these rotated-electron configurations and the local pseudofermion, Yang holon, and HL spinon occupancy configurations that describe the same state. Again, the number of local $\alpha\nu$ pseudofermions belonging to $\alpha\nu$ branches with finite occupancy in the virtual state are the same for all occupancy configurations. As for the rotated electrons, the relative position of these quantum objects is different in each configuration. Thus, when under a specific ground-state - excited-energy-eigenstate transition a $\alpha\nu$ pseudofermion or hole moves around the lattice ring, it scatters the same $\alpha'\nu'$ pseudofermion or hole scattering centers, but in

different order for different occupancy configurations. However, it is required that the phase factor $e^{iQ_{\alpha\nu}(q_j)}$ acquired by the $\alpha\nu$ pseudofermion or hole should be the same, independently of that order. This implies the commutativity of the S -matrix factors $S_{\alpha\nu, \alpha'\nu'}(q_j, q_{j'})$ in the overall S matrix $S_{\alpha\nu}(q_j)$ whose expression is given in Eq. (28). Such commutativity follows from the elementary S matrices $S_{\alpha\nu, \alpha'\nu'}(q_j, q_{j'})$ given in Eq. (29) being simple phase factors, instead of matrices of dimension larger than one. This seems to be inconsistent with all energy eigenstates being described by occupancy configurations which, besides $c0$ pseudofermions, involve finite-spin $1/2$ spinons and η -spin $1/2$ holons [4]. Indeed, the S matrix of finite- η -spin or spin quantum objects is a matrix of dimension larger than one [33, 34]. However, in spite of these finite- η -spin and finite-spin objects, due to symmetry the system self organizes in such a way that, in addition to the η -spin-less and spin-less $c0$ pseudofermions, the scatterers and scattering centers are the η -spin zero 2ν -holon composite $c\nu$ pseudofermions and spin zero 2ν -spinon composite $s\nu$ pseudofermions. Symmetry requirements also imply that the η -spin $1/2$ Yang holons and spin $1/2$ HL spinons [4, 5] are purely scatter-less objects, as discussed in the following.

Thus, it is the η -spin-neutral (and spin-neutral) character of the 2ν -holon (and 2ν -spinon) composite pseudofermion scatterers and scattering centers and the η -spin-less and spin-less character of the $c0$ pseudofermion scatterers and scattering centers which is behind the dimension of their S matrix $S_{\alpha\nu}(q_j)$ given in Eq. (28). Interestingly, in the pseudofermion scattering theory the relation of the composite $c\nu$ pseudofermion (and $s\nu$ pseudofermion) scatterers and scattering centers to the holons (and spinons) has similarities with that of the composite physical particles to the quarks in chromodynamics [35]. Within the latter theory the quarks have color but all quark-composite physical particles are color-neutral. Here the holons (and spinons) have η spin $1/2$ (and spin $1/2$) but the 2ν -holon (and 2ν -spinon) composite pseudofermion scatterers and scattering centers are η -spin-neutral (and spin-neutral).

In turn, the holon-holon and spinon-spinon S matrices of the conventional spinon-holon scattering theory of Refs. [30, 31, 32] do not have the above commutative properties. Indeed, within that theory the scatterers and scattering centers have η -spin $1/2$ or spin $1/2$ and for initial ground states with densities in the ranges $0 < n < 1$ and $0 < m < n$ many one-particle scattering states do not refer to energy eigenstates [33]. (The above mentioned requirement for commutative factorization of the S matrix applies when the one-particle scattering states belong to many-particle energy eigenstates and the scatterers and scattering centers are η -spin-neutral and/or spin-neutral.) On the other hand, for the $n = 1$ and $m = 0$ initial ground state considered in Refs. [30, 31, 32] the one-particle scattering states of the conventional spinon-holon representation of these references belong to many-particle energy eigenstates [33]. However, since the scatterers and scattering centers of that theory have η -spin $1/2$ or spin $1/2$, instead of the commutative factorization it is required that the S matrix has a YBE like factorization, as the BA bare S matrix of the original spin $1/2$ electrons.

D. SCATTERERS, SCATTERING CENTERS, AND SYMMETRY

The three generators of both the η -spin and spin $SU(2)$ algebras commute with the electron - rotated-electron unitary operator [4, 5]. As a result, the symmetry transformations of the $\alpha\nu$ pseudofermions, $\pm 1/2$ Yang holons, and $\pm 1/2$ HL spinons play an important role in their scattering properties.

For initial ground states with densities such that $0 < n < 1$ and $0 < m < n$ there is only finite occupancy for $c0$ pseudofermions and holes, $s1$ pseudofermions and holes, $+1/2$ Yang holons, and $+1/2$ HL spinons. The corresponding occupation numbers read $N_{c0} = N$, $N_{c0}^h = [N_a - N]$, $N_{s1} = N_\downarrow$, $N_{s1}^h = [N_\uparrow - N_\downarrow]$, $L_{c,+1/2} = [N_a - N]$, and $L_{s,+1/2} = [N_\uparrow - N_\downarrow]$. However, that for such ground states $N_{c0}^h = L_{c,+1/2}$ and $N_{s1}^h = L_{s,+1/2}$ does not imply that the $c0$ pseudofermion holes and the $s1$ pseudofermion holes are the same quantum objects as the $+1/2$ Yang holons and $+1/2$ HL spinons, respectively. For instance, the $[N_a - N]$ $+1/2$ Yang holons have finite η spin $1/2$ and the $[N_\uparrow - N_\downarrow]$ $+1/2$ HL spinons finite spin $1/2$ and both these objects are dispersion-less and have zero momentum and energy. In contrast, the $c0$ pseudofermion holes are η -spin-less and spin-less and the $s1$ pseudofermion holes have spin zero and are η -spin-less and both these types of pseudofermion holes have a momentum-dependent energy dispersion, $\epsilon_{c0}(\bar{q})$ and $\epsilon_{s1}(\bar{q})$, respectively. The ground-state canonical-momentum distributions of the $[N_a - N]$ $c0$ pseudofermion holes and $[N_\uparrow - N_\downarrow]$ $s1$ pseudofermion holes correspond to compact domains such that $2k_F < |\bar{q}| < \pi$ and $k_{F\downarrow} < |\bar{q}'| < k_{F\uparrow}$, respectively. Moreover, while under a ground-state - excited-energy-eigenstate transition the $c0$ and $s1$ pseudofermion-hole discrete canonical-momentum values \bar{q}_j and \bar{q}'_j acquire a shift given by $Q_{c0}(q_j)/L$ and $Q_{s1}(q'_j)/L$, respectively, the momentum zero of the $+1/2$ Yang holons and $+1/2$ HL spinons remains unchanged. Also the momentum values π and zero of the $-1/2$ Yang holons and $-1/2$ HL spinons, respectively, remain unchanged under such a transition.

The form of the scattering part of the overall phase shift (16), Eq. (4), reveals that the value of such a phase-shift functional is independent of the changes in the occupation numbers of the $\pm 1/2$ Yang holons and $\pm 1/2$ HL spinons. Thus, these objects are not scattering centers. Moreover, they are not scatterers, once their momentum values remain unchanged under the transition from the “in” state to the “out” state (excited energy eigenstate). Such a scatter-less

character of the $\pm 1/2$ Yang holons and $\pm 1/2$ HL spinons is related to the above symmetry. Indeed, the $\pm 1/2$ Yang holons (and $\pm 1/2$ HL spinons) are created and annihilated by the η -spin (and spin) off-diagonal generators. Since these generators commute with the electron - rotated-electron unitary operator, they have the same U -independent expressions both in terms of electronic and rotated-electronic operators [4, 5]. That the $\pm 1/2$ Yang holons and $\pm 1/2$ HL spinons are neither scatterers nor scattering centers is consistent with that property.

In contrast, since the pseudofermions and pseudofermion holes are not in general invariant under the electron - rotated-electron unitary transformation [4, 6], their creation and annihilation operators have U -dependent expressions in terms of electronic operators. That these quantum objects are scatterers and scattering centers is consistent with such a property. Moreover, these objects are not transformed by the η -spin and spin generators. This latter symmetry is behind the $c0$ pseudofermion being a η -spin-less and spin-less object and for $\nu > 0$ the $\alpha\nu$ pseudofermions being η -spin ($\alpha = c$) and spin ($\alpha = s$) singlet 2ν -holon and 2ν -spinon composite objects, respectively.

The particular case of the invariance under the electron - rotated-electron unitary transformation of $\alpha\nu \neq c0$, $s1$ pseudofermions of limiting canonical-momentum values $\bar{q} = \pm q_{\alpha\nu}^0$ and the corresponding separation of these objects into 2ν independent holons ($\alpha\nu = c\nu$) or spinons ($\alpha\nu = s\nu$) and $\alpha\nu \neq c0$, $s1$ FP (*Fermi*-point) current scattering centers is discussed below in Sec. IV-C.

E. PSEUDOFERMION S MATRIX AND THE FINITE-ENERGY SPECTRAL PROPERTIES

The simple form of the pseudofermion and hole S matrix renders the pseudofermion description particularly suitable for the study of the unusual finite-energy spectral properties of the model. Such a simplicity of the S matrix form results in part from all “in” and “out” states being energy eigenstates. Indeed, the latter states can be expressed as direct products of the “in” and “out” one-pseudofermion scattering states of the theory. Fortunately, this allows the use of Lehmann representations for the study of the spectral functions [14, 15, 16]. Moreover, the simple form of the pseudofermion and hole S matrix simplifies the evaluation of the spectral-function matrix elements between the initial ground state and the exact excited energy eigenstates, which are the “out” states. Indeed, the anticommutation relations (15) can be expressed solely in terms of the difference $\bar{q} - \bar{q}'$ of the two pseudofermion momenta and the S matrix $S_{\alpha\nu}(q)$ given in Eq. (28) of the pseudofermion associated with the excited energy eigenstate as follows,

$$\{f_{\bar{q}, \alpha\nu}^\dagger, f_{\bar{q}', \alpha'\nu'}\} = \frac{\delta_{\alpha\nu, \alpha'\nu'}}{N_{\alpha\nu}^*} [S_{\alpha\nu}(q)]^{1/2} e^{-i(\bar{q}-\bar{q}')/2} \frac{\Im[S_{\alpha\nu}(q)]^{1/2}}{\sin[(\bar{q} - \bar{q}')/2]}, \quad (32)$$

and the anticommutators between two $\alpha\nu$ pseudofermion creation or annihilation operators vanish. This reveals that the S matrix $S_{\alpha\nu}(q_j)$ fully controls the pseudofermion anticommutators. Since within the PDT these anticommutators determine the value of the matrix elements between the ground state and the excited energy eigenstates [14, 15, 16], it follows that the S matrix $S_{\alpha\nu}(q_j)$ controls the finite-energy spectral properties. If it had dimension larger than one, the pseudofermion algebra would be much more involved, for the pseudofermion anticommutators would also be matrices of dimension larger than one. The problem of the evaluation of the spectral-function matrix elements between energy eigenstates simplifies for the pseudofermion representation because the PS subspaces associated with a given one- or two-electron spectral function can be expressed in terms of direct products corresponding to each of the $\alpha\nu$ branch quantum-number occupancy configurations of branches with finite pseudofermion occupancy [14, 15, 16]. For these matrix elements the direct product is associated with the commutative factorization of the S matrix (28) in terms of the elementary S matrices $S_{\alpha\nu, \alpha'\nu'}(q_j, q_{j'})$, Eq. (29). Therefore, the use of the pseudofermion description considerably simplifies the study of the exotic quantum-liquid finite-energy spectral properties [15, 16, 23, 24, 25].

The PDT of Refs. [14, 15, 16] confirms that the overall $\alpha\nu$ pseudofermion and hole phase-shift functional (16) associated with the S matrix $S_{\alpha\nu}(q_j)$ of Eq. (28) fully controls the one- and two-electron spectral properties through the pseudofermion anticommutators. An one- or two-electron excitation $\mathcal{O}^\dagger|GS\rangle$ is contained in a well defined direct sum (14) of CPHS ensemble subspaces and is a superposition of excited energy eigenstates. Each of these states is described by a set of deviations $\{\Delta N_{\alpha\nu}(q_j)\}$, Eq. (5), involving a finite number of scattering centers. Such deviations obey the sum rules (18) and (19) of Ref. [14] and the selection rules given in Eq. (21) of the same reference.

One can consider that the order of the direct sum (14) of CPHS ensemble subspaces associated with a given one- or two-electron excitation is that of the increasing values of the minimum energy $\omega_0(D_r, S_r)$, Eq. (13), of each CPHS ensemble subspace relative to the initial ground state. For fixed value of S_r and D_r , these energy values are such that,

$$\omega_0(0, S_r) < \omega_0(1, S_r) < \omega_0(2, S_r) < \dots, \quad (33)$$

and

$$\omega_0(D_r, 0) < \omega_0(D_r, 1) < \omega_0(D_r, 2) < \dots, \quad (34)$$

respectively. Note that for fixed values of electronic density n and spin density m , the ordering of the energies $\omega_0(D_r, S_r)$ is well defined and such that,

$$\omega_0(0, 0) < \min\{\omega_0(0, 1), \omega_0(1, 0)\} < \dots \quad (35)$$

An important property is that for a given energy range $0 < \Delta E < \omega_0$ the number of CPHS ensemble subspaces of the direct sum (14) is finite. Moreover, as each energy $\omega_0(D_r, S_r)$ is reached, an increasing number of channels open up that correspond to CPHS ensemble subspaces of increasing energy. By new channels we mean here $\alpha\nu$ pseudofermion branches of increasing ν value. Thus, for a given energy range $0 < \Delta E < \omega_0$ there is a well defined set of pseudofermion S matrices, whose “in” and “out” asymptote one-pseudofermion scattering states correspond to the set of excited energy eigenstates spanning the direct sum (14) of CPHS ensemble subspaces of energy smaller than or equal to ω_0 . The pseudofermion S matrices belonging to that finite set control the spectral properties for energies in the above range $0 < \Delta E < \omega_0$.

The pseudofermion representation is valid for $L \gg 1$. Indeed, the expressions of all quantities derived by use of that representation are physical up to first order in $1/L$. Thus, for the study of some properties one can replace the discrete bare-momentum q_j and canonical-momentum \bar{q}_j by a continuum bare-momentum variable q and canonical-momentum variable \bar{q} , respectively. The ground-state is then described by a $c0$ occupancy for $|q| < 2k_F$ and unoccupancy for $2k_F < |q| < \pi$ and a $s1$ occupancy for $|q| < k_{F\downarrow}$ and unoccupancy for $k_{F\downarrow} < |q| < k_{F\uparrow}$. (All remaining $\alpha\nu$ pseudofermion bands are unoccupied for the ground state.) Thus, in the continuum momentum limit, the global canonical-momentum shift $Q_{\alpha\nu}(q)/L = Q_{\alpha\nu}^0/L + Q_{\alpha\nu}^\Phi(q)/L$ contributes to the spectral properties mainly through the $\alpha\nu = c0, s1$ branches for q values in the vicinity of the *Fermi points* $\pm q_{\alpha\nu}^0$, as confirmed by the studies of Refs. [14, 15, 16]. However, often such a limit must be taken in the end of the calculations. Otherwise, one would lose the information contained in the overall pseudofermion or hole phase shifts studied below, which correspond to canonical-momentum shifts, $Q_{\alpha\nu}(q)/L$, of the order of $1/L$.

IV. THE PSEUDOFERMION SCATTERING THEORY: PSEUDOFERMION PHASE SHIFTS

In this section we study the $\alpha\nu$ pseudofermion phase shifts associated with the S matrix introduced above.

A. PHASE-SHIFT DEFINITION

The effective $\alpha\nu$ lattices have the same length L as the rotated-electron and electronic lattice [6]. As above, our analysis refers to periodic boundary conditions and $L \gg 1$. Depending on the asymptote coordinate choice, there are two possible definitions for the $\alpha\nu$ pseudofermion phase shifts associated with the S matrix $S_{\alpha\nu}(q_j)$ given in Eq. (28). The choice of either definition is a matter of taste and the uniquely defined quantity is the S matrix. The two choices of asymptote coordinates for the $\alpha\nu$ pseudofermion or $\alpha\nu$ pseudofermion hole correspond to $x \in (-L/2, +L/2)$ and $x \in (0, +L)$.

If when moving around the lattice ring the $\alpha\nu$ pseudofermion (or hole) departures from the point $x = -L/2$ and arrives to $x = L/2$, one finds that,

$$\lim_{x \rightarrow L/2} \bar{q}x = qx + Q_{\alpha\nu}^0/2 + Q_{\alpha\nu}^\Phi(q)/2 = qx + \delta_{\alpha\nu}(q), \quad (36)$$

where

$$\delta_{\alpha\nu}(q) = Q_{\alpha\nu}(q)/2. \quad (37)$$

For this asymptote coordinate choice, $\delta_{\alpha\nu}(q)$ is the overall $\alpha\nu$ pseudofermion or hole phase shift whose value is given only to within addition of an arbitrary multiple of π . Moreover, from analysis of Eqs. (4) and (16) it follows that $\pi \Phi_{\alpha\nu, \alpha'\nu'}(q_j, q_{j'})$ is an elementary two-pseudofermion phase shift. This phase-shift definition corresponds to that used in standard quantum non-relativistic scattering theory [34], such that the S matrix $S_{\alpha\nu}(q_j)$ given in Eq. (28) can be written as,

$$S_{\alpha\nu}(q_j) = e^{i2\delta_{\alpha\nu}(q_j)}, \quad j = 1, 2, \dots, N_{\alpha\nu}^*. \quad (38)$$

The factor 2 appearing in the exponential argument of Eq. (38) corresponds to the usual form of the S matrix for that theory. In reference [36] it is found that such a phase-shift definition is consistent with an exact theorem due to Fumi [37]. Note that for the phase-shift definition (37), $Q_{\alpha\nu}^0/2 = 0, \mp\pi/2$ corresponds to the scatter-less term $-l\pi/2$

of the three-dimensional partial-wave problem of angular momentum l [34, 37]. Although the angular momentum vanishes in 1D, the scatter-less phase shift (17) plays a similar role. Here we follow the definition of the standard quantum non-relativistic scattering theory and choose the overall $\alpha\nu$ pseudofermion phase shift definition $Q_{\alpha\nu}(q)/2$ associated with Eq. (36).

In turn, if when moving around the lattice ring the $\alpha\nu$ pseudofermion (or hole) departs from the point $x = 0$ and arrives to $x = L$, one finds that,

$$\lim_{x \rightarrow L} \bar{q}x = qx + Q_{\alpha\nu}^0 + Q_{\alpha\nu}^\Phi(q) = qx + Q_{\alpha\nu}(q), \quad (39)$$

where q refers to the initial ground state. For this asymptote coordinate choice, $Q_{\alpha\nu}(q)$ is the overall $\alpha\nu$ pseudofermion (or hole) phase shift whose value is given only to within addition of an arbitrary multiple of 2π and $2\pi \Phi_{\alpha\nu, \alpha'\nu'}(q_j, q_{j'})$ is an elementary two-pseudofermion phase shift.

The studies of Ref. [33] reveal that the overall pseudofermion phase-shift choice $Q_{\alpha\nu}(q) = Q_{\alpha\nu}^0 + Q_{\alpha\nu}^\Phi(q)$ associated with the asymptote condition (39) corresponds to a generalization of the conventional phase-shift definition previously used in the BA literature for the particular case of the $n = 1$ and $m = 0$ initial ground state. (All the discussions and analysis presented below in this paper also apply to the phase-shift definition $Q_{\alpha\nu}(q) = Q_{\alpha\nu}^0 + Q_{\alpha\nu}^\Phi(q)$, provided that the $\alpha\nu$ phase shifts $\delta_{\alpha\nu}(q) = Q_{\alpha\nu}(q)/2$ are multiplied by two.)

B. THE TWO-PSEUDOFERMION PHASE SHIFTS: BARE-MOMENTUM DEPENDENCE AND LEVINSON'S THEOREM

The scattering $\alpha\nu$ pseudofermion or hole phase-shift $Q_{\alpha\nu}^\Phi(q)/2$ given in Eq. (4) results from the set of elementary two-pseudofermion scattering events associated with the transition from the “in” state to the “out” state (excited energy eigenstate). In contrast, the transition from the ground state to the “in” state has no scattering character and leads to the scatter-less phase-shift $Q_{\alpha\nu}^0/2 = 0, \mp\pi/2$ of Eq. (17).

The bare-momentum two-pseudofermion phase shifts $\Phi_{\alpha\nu, \alpha'\nu'}(q, q')$ in units of π are related to the rapidity two-pseudofermion phase shifts $\bar{\Phi}_{\alpha\nu, \alpha'\nu'}(r, r')$ by Eq. (6). The latter phase shifts are defined by the integral equations (A1)-(A13) of Ref. [6]. In Appendix B we provide a set of simplified equations which define the rapidity elementary two-pseudofermion phase shifts $\bar{\Phi}_{\alpha\nu, \alpha'\nu'}(r, r')$ in the limit $m \rightarrow 0$. Furthermore, in that Appendix we also provide closed form expressions valid in the specific limit $m \rightarrow 0$ and $n \rightarrow 1$.

In figures 1-6 we plot the two-pseudofermion phase shifts which contribute most to these properties as a function of the bare-momenta q and q' for electronic density $n = 0.59$, spin density $m \rightarrow 0$, and $U/t \rightarrow 0, U/t = 0.3, 4.9, 100$. (The electronic density value $n = 0.59$ and the value $U/t = 4.9$ are those used in Ref. [24] for the description of the TCNQ photoemission dispersions observed in the quasi-1D organic compound TTF-TCNQ.) Analytical expressions valid for $U/t \rightarrow 0$ as $m \rightarrow 0$ of the same two-pseudofermion phase shifts are provided in Appendix C.

Note that the function $\Phi_{c0, c1}(q, q')$ plotted in Fig. 5 can have values in the domain $\Phi_{c0, c1}(q, q') \in (-1, 1)$. Thus, within the standard quantum non-relativistic scattering theory phase shift definition given in Eq. (36), the corresponding phase shift $\pi \Phi_{c0, c1}(q, q')$ has values in the domain $\pi \Phi_{c0, c1}(q, q') \in (-\pi, \pi)$. Note that the width of this domain is 2π , whereas the definition (39) would lead to a domain width of 4π .

For $m \rightarrow 0$ the two-pseudofermion phase shifts have a similar qualitative behavior for a large domain of U/t values corresponding to $U/t > 4$. Thus, in order to obtain further information about the q and q' dependence of such phase shifts, it is useful to derive an analytical expansion in t/U for $m \rightarrow 0$, which provides the exact behavior for $U/t \gg 1$. By use of manipulations similar to those reported in Appendix C, we find that for $m \rightarrow 0$ and $U/t \gg 1$ the elementary two-pseudofermion phase shifts plotted in Figs. 1-6 can be written as follows,

$$\pi \Phi_{c0, c0}(q, q') = -\frac{\pi(\xi_0 - 1/\xi_0)}{2} \left(\frac{\sin(q) - \sin(q')}{2 \sin(\pi n)} \right) + \pi \left[\frac{(\xi_0 + 1/\xi_0)}{2} - 1 \right] \frac{\sin(q')}{\sin(\pi n)}, \quad (40)$$

$$\pi \Phi_{c0, s1}(q, q') = \frac{1}{2} \arctan \left(\sinh \left(-\frac{2\pi t}{U} \sin(q) + \operatorname{arcsinh} \left(\tan \left(\frac{q'}{n} \right) \right) \right) \right) + \frac{(\xi_0 - 1)}{4} \frac{2q'}{n}, \quad (41)$$

$$\begin{aligned} \pi \Phi_{s1, c0}(q, q') &= -\frac{1}{2} \arctan \left(\sinh \left(\operatorname{arcsinh} \left(\tan \left(\frac{q}{n} \right) \right) - \frac{2\pi t}{U} \sin(q') \right) \right); \quad q \neq \pm k_F \\ &= -\frac{\operatorname{sgn}(q)\pi}{2\sqrt{2}}; \quad q = \pm k_F, \end{aligned} \quad (42)$$

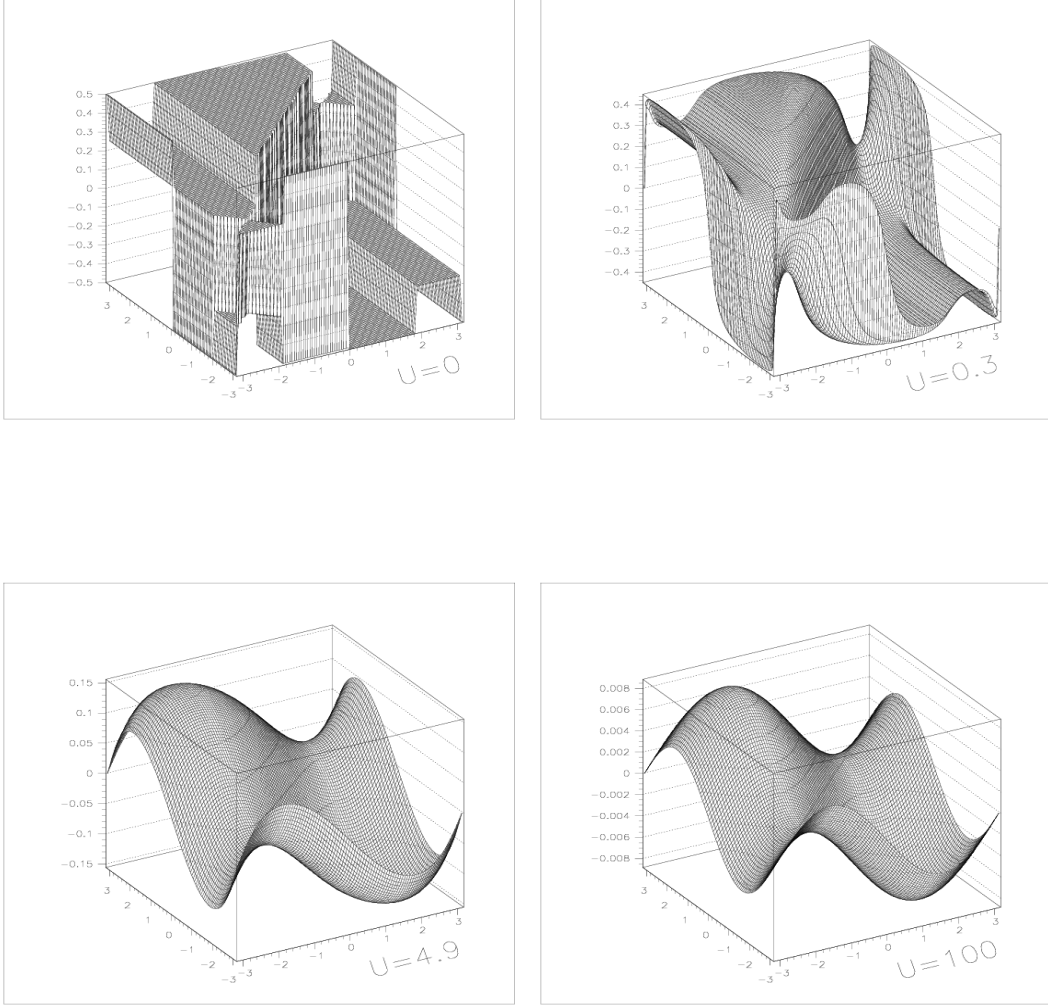


FIG. 1: The elementary two-pseudofermion phase shift $\Phi_{c0, c0}(q, q')$ in units of π as a function of q and q' for $n = 0.59$, $m = 0$, and (a) $U/t \rightarrow 0$, (b) $U/t = 0.3$, (c) $U/t = 4.9$, and (d) $U/t = 100$. The bare-momentum values q and q' correspond to the right and left axis of the figures, respectively.

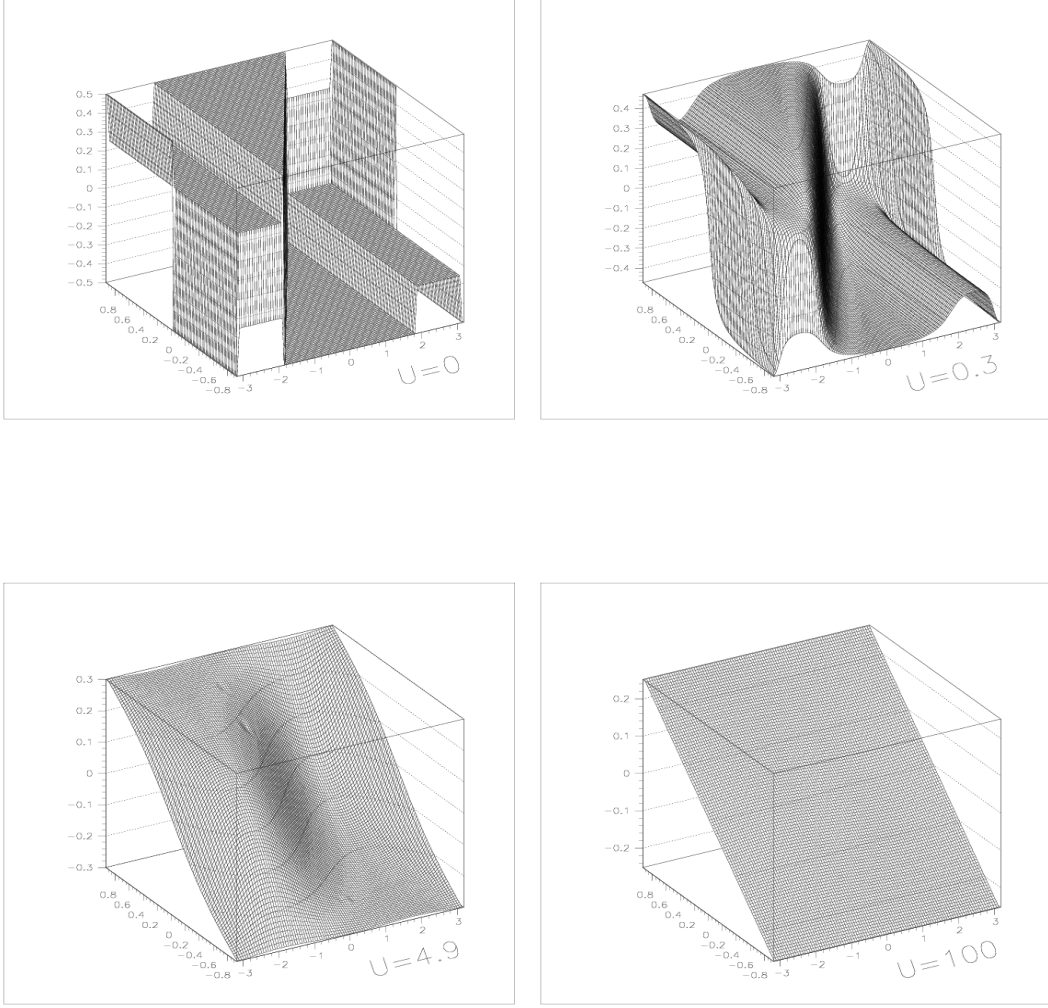


FIG. 2: The elementary two-pseudofermion phase shift $\Phi_{c0,s1}(q, q')$ in units of π as a function of q and q' for $n = 0.59$, $m = 0$, and (a) $U/t \rightarrow 0$, (b) $U/t = 0.3$, (c) $U/t = 4.9$, and (d) $U/t = 100$. The bare-momentum values q and q' correspond to the right and left axis of the figures, respectively.

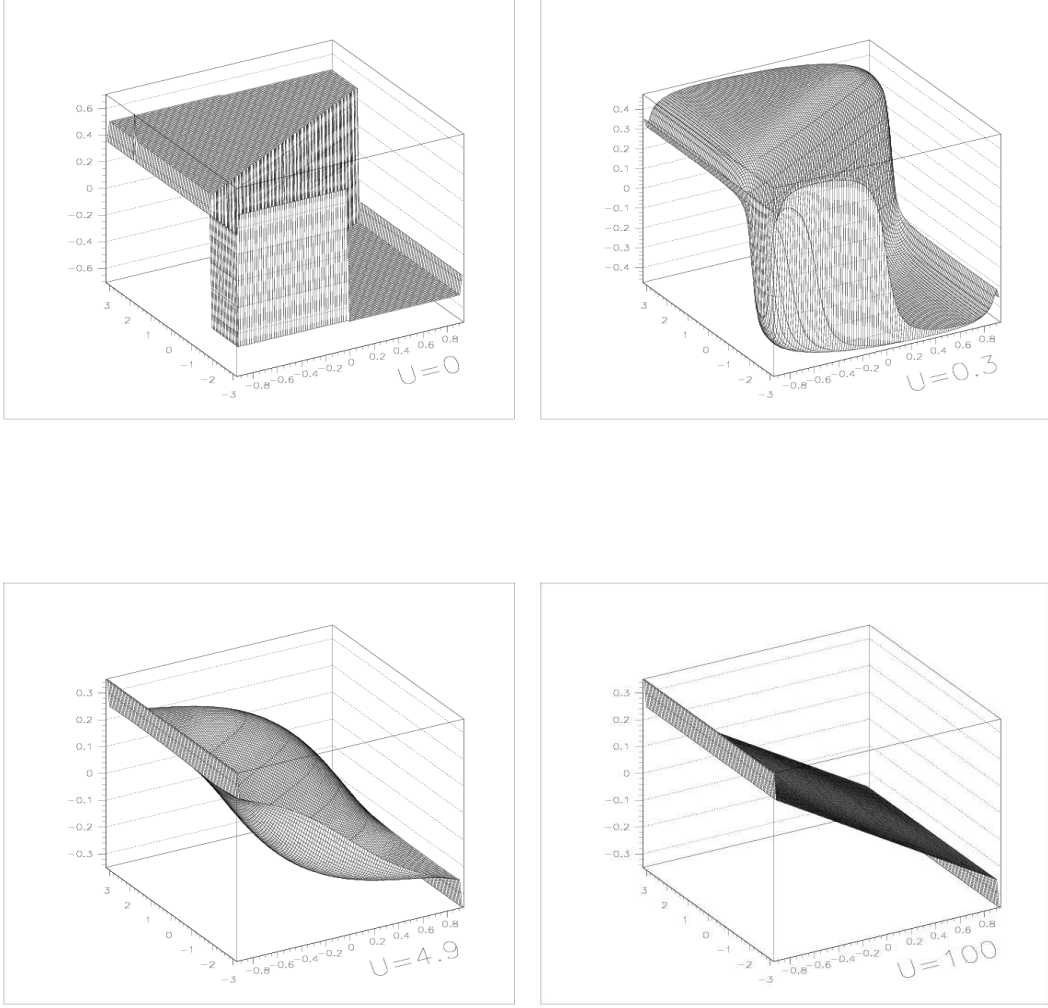


FIG. 3: The elementary two-pseudofermion phase shift $\Phi_{s1,c0}(q, q')$ in units of π as a function of q and q' for $n = 0.59$, $m = 0$, and (a) $U/t \rightarrow 0$, (b) $U/t = 0.3$, (c) $U/t = 4.9$, and (d) $U/t = 100$. The bare-momentum values q and q' correspond to the right and left axis of the figures, respectively.

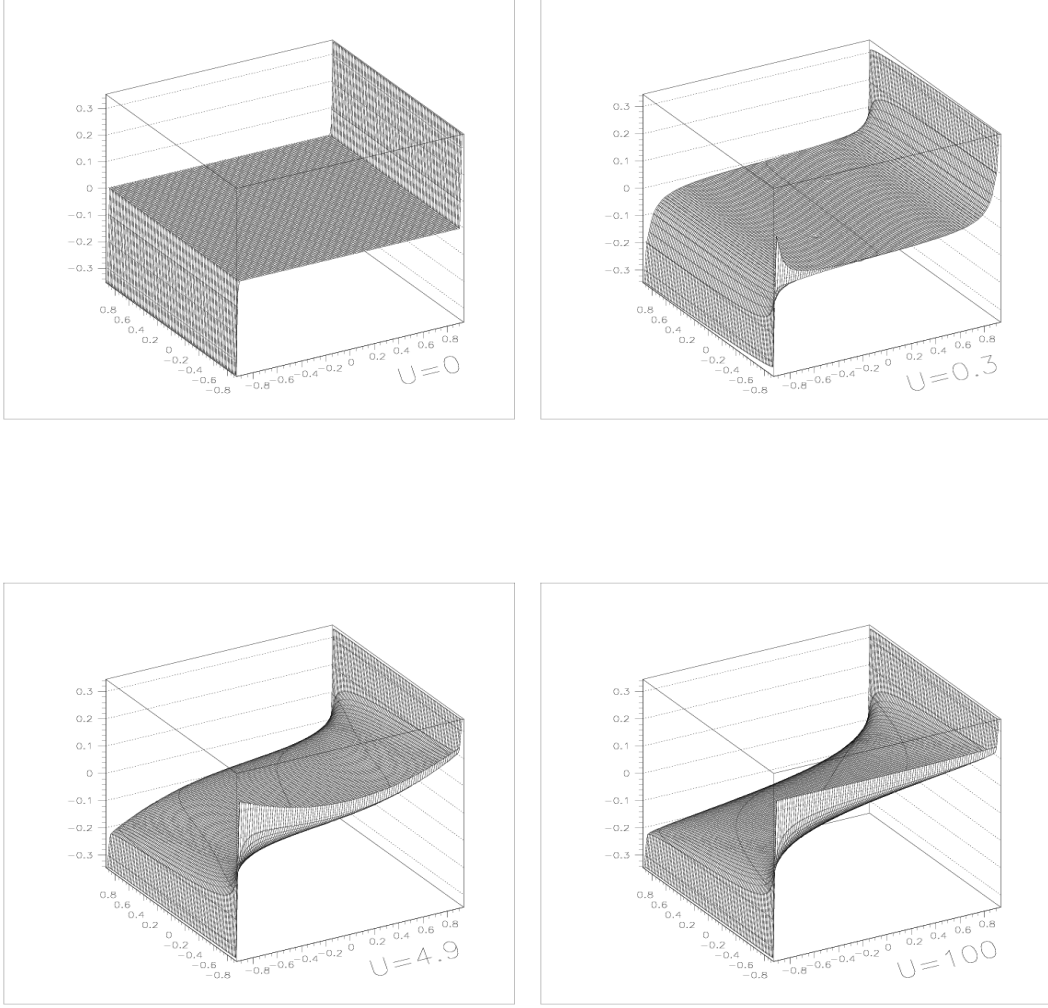


FIG. 4: The elementary two-pseudofermion phase shift $\Phi_{s1, s1}(q, q')$ in units of π as a function of q and q' for $n = 0.59$, $m = 0$, and (a) $U/t \rightarrow 0$, (b) $U/t = 0.3$, (c) $U/t = 4.9$, and (d) $U/t = 100$. The bare-momentum values q and q' correspond to the right and left axis of the figures, respectively.

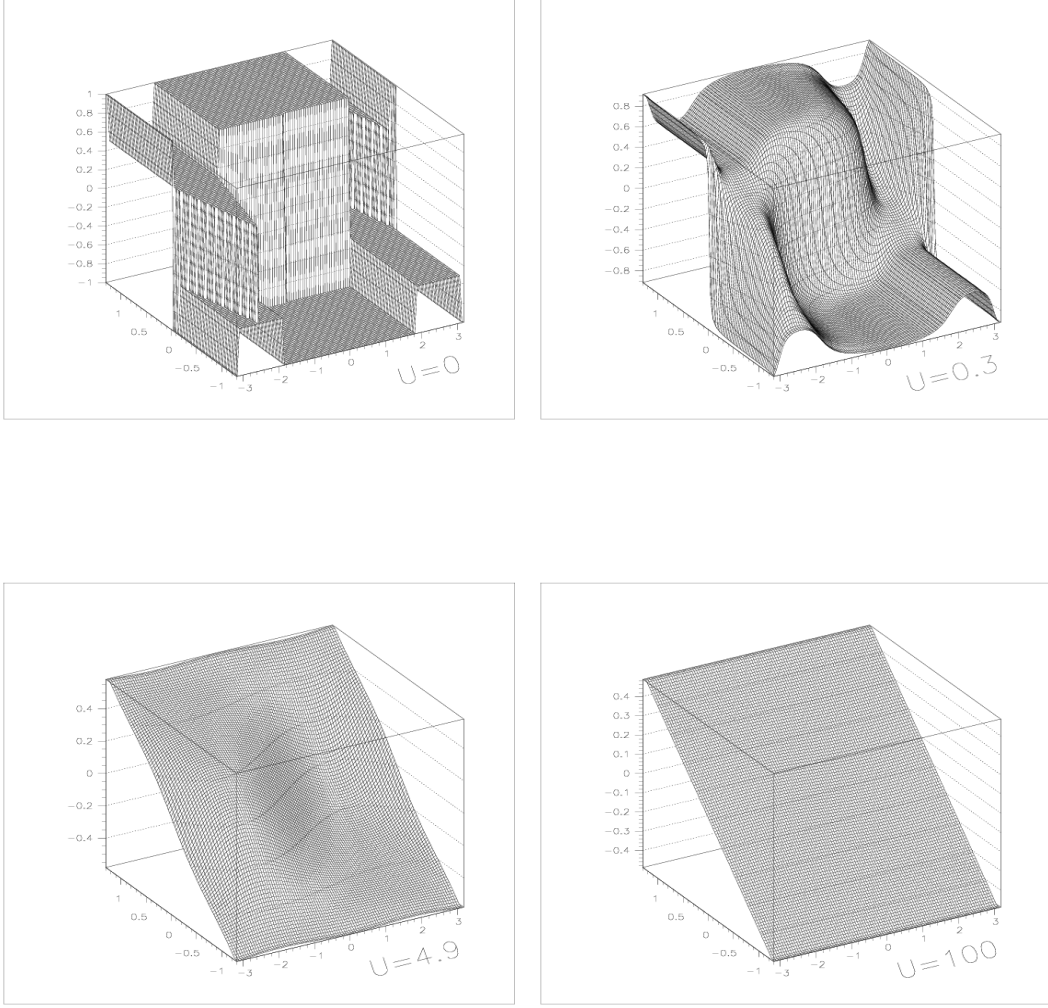


FIG. 5: The elementary two-pseudofermion phase shift $\Phi_{c0, c1}(q, q')$ in units of π as a function of q and q' for $n = 0.59$, $m = 0$, and (a) $U/t \rightarrow 0$, (b) $U/t = 0.3$, (c) $U/t = 4.9$, and (d) $U/t = 100$. The bare-momentum values q and q' correspond to the right and left axis of the figures, respectively.

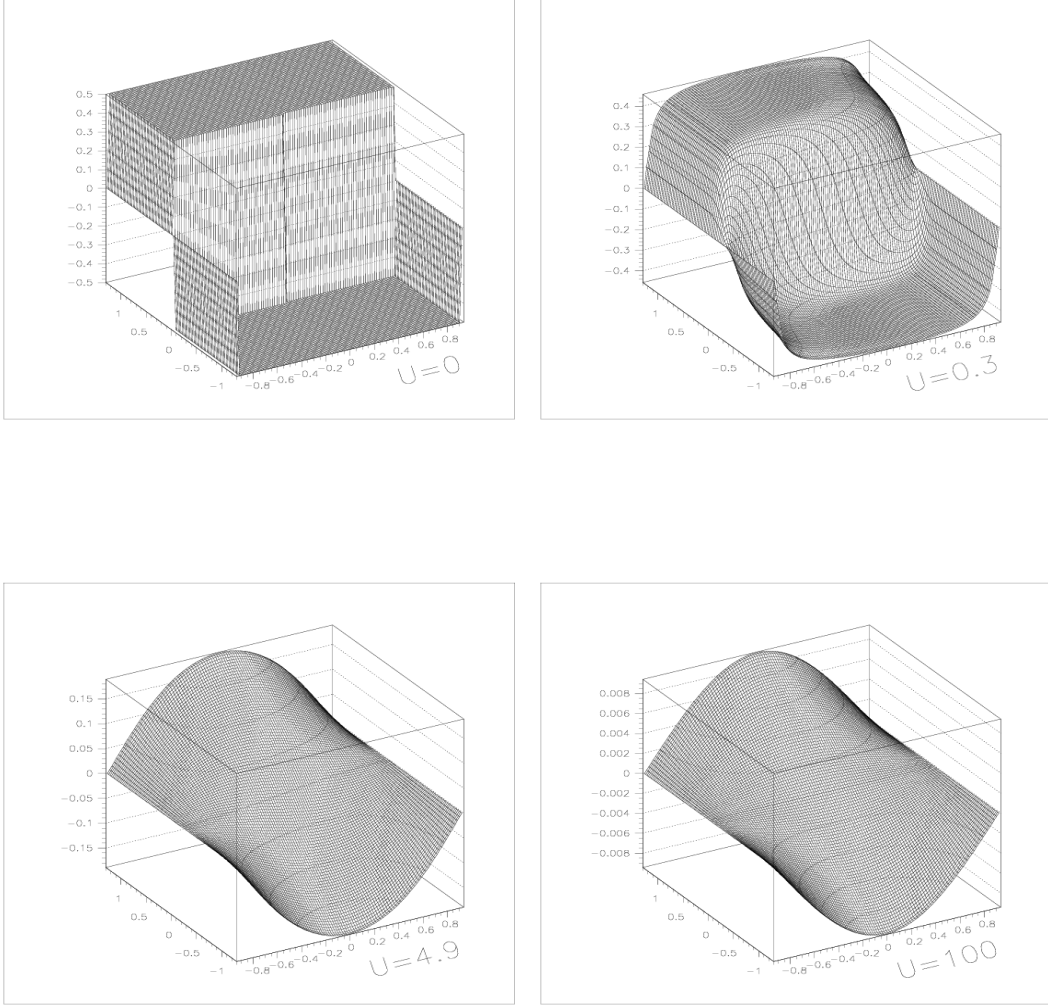


FIG. 6: The elementary two-pseudofermion phase shift $\Phi_{s1, c1}(q, q')$ in units of π as a function of q and q' for $n = 0.59$, $m = 0$, and (a) $U/t \rightarrow 0$, (b) $U/t = 0.3$, (c) $U/t = 4.9$, and (d) $U/t = 100$. The bare-momentum values q and q' correspond to the right and left axis of the figures, respectively.

$$\begin{aligned}
\pi \Phi_{s1, s1}(q, q') &= \int_0^\infty d\omega \frac{\sin\left(\omega \frac{2}{\pi} [\text{arc sinh}\left(\tan\left(\frac{q}{n}\right)\right) - \text{arc sinh}\left(\tan\left(\frac{q'}{n}\right)\right)]\right)}{\omega(1 + e^{2\omega})} \\
&+ \frac{t}{U} \frac{\sin(\pi n)}{2} \frac{2q'}{n} \cos\left(\frac{q}{n}\right); \quad q \neq \pm k_F \\
&= \frac{\text{sgn}(q)\pi}{2\sqrt{2}}; \quad q = \pm k_F, \quad q' \neq \pm k_F \\
&= [\text{sgn}(q)]\pi \left(\frac{3}{2\sqrt{2}} - 1\right); \quad q = q' = \pm k_F, \quad ,
\end{aligned} \tag{43}$$

$$\pi \Phi_{c0, c1}(q, q') = \arctan\left(-\frac{4t}{U} \sin(q) + \tan\left(\frac{q'}{2\delta}\right)\right) + \frac{(\xi_0 - 1)}{2} \frac{q'}{\delta}, \tag{44}$$

and

$$\pi \Phi_{s1, c1}(q, q') = \frac{t}{U} \sin(\pi\delta) \frac{2q'}{\delta} \cos\left(\frac{q}{n}\right), \tag{45}$$

respectively. In these equations $\delta = 1 - n$ and ξ_0 is the interaction-dependent parameter which for zero spin density appears in the expressions of the low-energy quantities [7, 8, 9, 10, 11, 12, 13, 21, 22, 38, 39]. For instance, it is defined in Eq. (74) of Ref. [39]. In the above $U/t \gg 1$ two-pseudofermion expansions (40), (41), and (44), ξ_0 should be replaced by its first-order expansion in t/U ,

$$\xi_0 = 1 + \frac{4t}{\pi U} \ln 2 \sin(\pi n); \quad U/t \gg 1. \tag{46}$$

Note that the t/U leading-order term of the quantity $(\xi_0 + 1/\xi_0)$ appearing in Eq. (40) is of second order. However, since the t/U second order term of the parameter ξ_0 , which is not given in Eq. (46), does not contribute to $(\xi_0 + 1/\xi_0)$, that leading-order term should be considered.

For $U/t = 100$ and $n = 0.59$ the bare-momentum dependence of the two-pseudofermion phase-shift analytical expansions (40)-(45) is similar to the exact dependence plotted in Figs. 1-6. In figure 7 such phase-shift expansions are plotted for $U/t = 4.9$ and $n = 0.59$. In spite that they reproduce the exact behavior for $U/t \gg 1$ only, note that their use for $U/t = 4.9$ is indeed a reasonably good approximation. This can be confirmed by comparison of the two-pseudofermion phase-shift expansions plotted in Fig. 7 with the corresponding exact phase-shift values plotted in Figs. 1-6. This result is consistent with the $U/t \gg 1$ physics being dominant from $U/t \approx 4$ until $U/t \rightarrow \infty$. It follows that the phase-shift analytical expressions (40)-(45) can be used as an approximation for the two-pseudofermion phase-shift expressions for such a range of U/t values, becoming an increasingly better approximation as U/t increases.

In order to confirm that our theory is consistent with the expected general properties of the standard scattering theory, let us check whether the phase shifts $\pi \Phi_{\alpha\nu, \alpha'\nu'}(q, q')$ associated with the elementary two-pseudofermion scattering events obey Levinson's Theorem [40]. Such a theorem just states that when in the reference frame of the scattering center the momentum of the scatterer tends to zero the phase shift is given by πN_b , where N_b is the number of bound states. In that frame the phase shift $\pi \Phi_{\alpha\nu, \alpha'\nu'}(q, q')$ reads $\pi \Phi_{\alpha\nu, \alpha'\nu'}(q - q', 0)$. Moreover, in our case there are no bound states and thus $N_b = 0$. Therefore, for the pseudofermion scattering theory Levinson's Theorem requires that,

$$\lim_{q-q' \rightarrow 0} \pi \Phi_{\alpha\nu, \alpha'\nu'}(q - q', 0) = 0. \tag{47}$$

The validity of this result is confirmed by suitable analysis of the integral equations (A1)-(A13) of Ref. [6], which reveals that $\pi \bar{\Phi}_{\alpha\nu, \alpha'\nu'}(r, r') = -\pi \bar{\Phi}_{\alpha\nu, \alpha'\nu'}(-r, -r')$. This result combined with the use of Eq. (6) and the odd character of the ground-state rapidity functions, such that $\Lambda_{\alpha\nu}^0(q) = -\Lambda_{\alpha\nu}^0(-q)$ [15], leads then to,

$$\pi \Phi_{\alpha\nu, \alpha'\nu'}(q, q') = -\pi \Phi_{\alpha\nu, \alpha'\nu'}(-q, -q'). \tag{48}$$

This latter symmetry implies that $\pi \Phi_{\alpha\nu, \alpha'\nu'}(q - q', 0)$ is a odd function of $q - q'$, what confirms the validity of the Levinson's Theorem (47).

Finally, let us address the relation of the pseudofermion phase shifts to the phase shifts considered in Refs. [38, 39]. It is straightforward to show that the elementary phase shifts defined by Eqs. (32)-(38) of Ref. [39] correspond to a particular case of the rapidity elementary two-pseudofermion phase shifts defined by the integral equations (A1)-(A13)

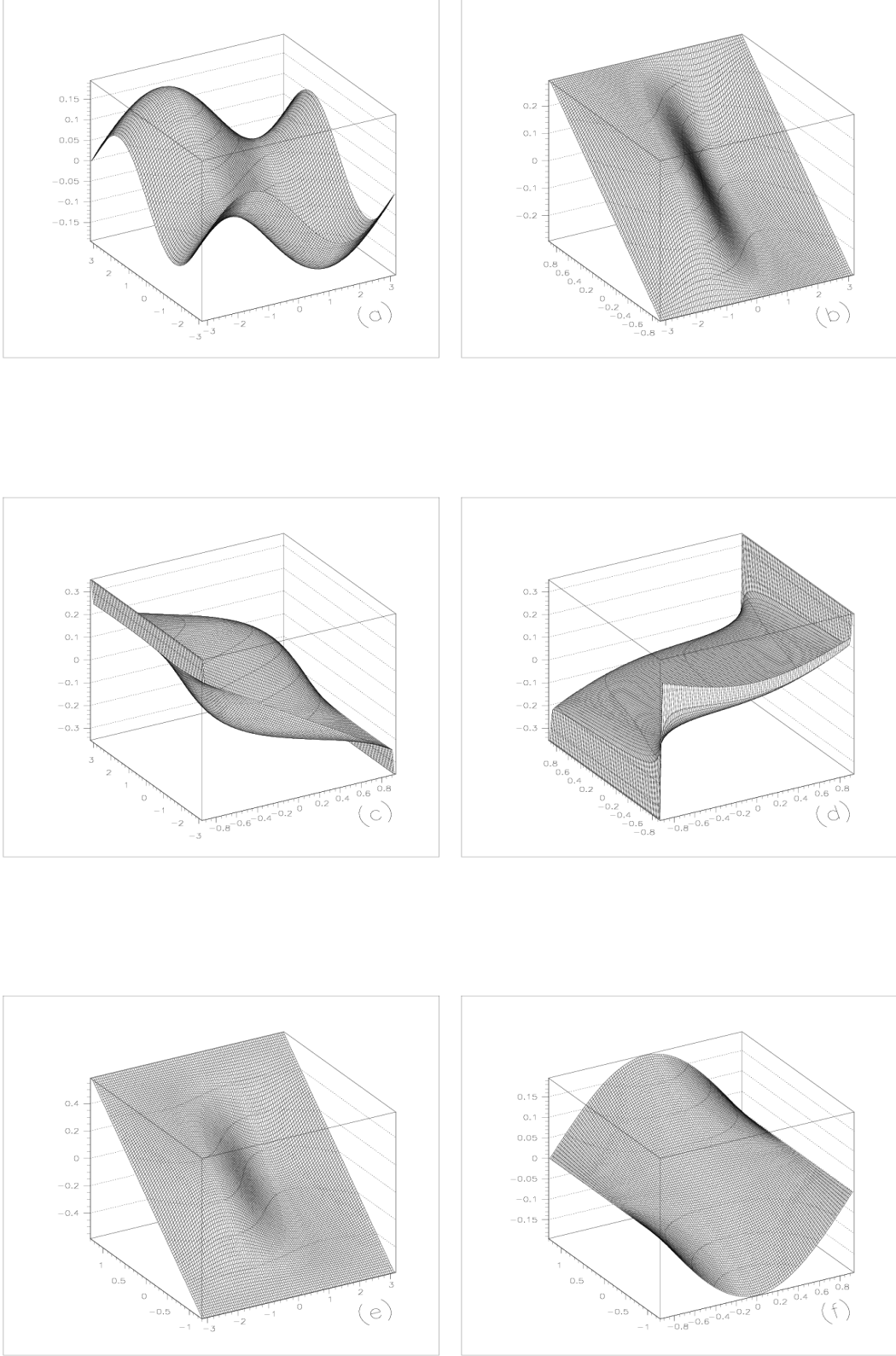


FIG. 7: The $U/t \gg 1$ two-pseudofermion phase-shift expansions (40)-(45) in units of π for (a) $\Phi_{c0,c0}(q, q')$, (b) $\Phi_{c0,s1}(q, q')$, (c) $\Phi_{s1,c0}(q, q')$, (d) $\Phi_{s1,s1}(q, q')$, (e) $\Phi_{c0,c1}(q, q')$, and (f) $\Phi_{s1,c0}(q, q')$, respectively, as a function of q and q' for $n = 0.59$ and $U/t = 4.9$. The bare-momentum values q and q' correspond to the right and left axis of the figures, respectively. For $U/t = 4.9$ the phase-shift large U/t expansions (40)-(45) are a reasonably good approximation for the corresponding exact phase shifts

of Ref. [6]. If one considers the PS subspace spanned by energy eigenstates such that $N_{\alpha\nu} = 0$ for the $\alpha\nu \neq c0, s1$ branches and $L_{\alpha, -1/2} = 0$ for $\alpha = c, s$, the general integral equations (A1)-(A13) of Ref. [6] reduce to the integral equations (32)-(38) of Ref. [39]. Thus, the phase shifts previously considered in Refs. [38, 39] correspond to a particular case of the general elementary two-pseudofermion phase shifts. In contrast to the interpretation of Refs. [38, 39], the scatterers and scattering centers associated with the phase shifts defined by Eqs. (32)-(38) of Ref. [39] are the pseudofermions and pseudofermion holes, rather than the corresponding pseudoparticles and pseudoparticle holes considered in that reference. Indeed, under the ground-state - excited-state transitions the discrete bare-momentum values of the pseudoparticles do not acquire the scattering momentum shift $Q_{\alpha\nu}^\Phi(q_j)/L$ given in Eq. (4). Instead, within the pseudoparticle representation the functional $Q_{\alpha\nu}^\Phi(q_j)$ appears in the energy spectrum, where it leads to f -function energy terms associated with the two-pseudoparticle residual interactions [6]. In turn, such interactions do not exist for the pseudofermion representation.

C. INVARIANCE UNDER THE ELECTRON - ROTATED-ELECTRON UNITARY TRANSFORMATION

In the previous section it was mentioned that the pseudofermions are not in general invariant under the electron - rotated-electron unitary transformation. However, the exception is for the $\alpha\nu \neq c0, s1$ branches as the canonical momentum \bar{q} approaches the limiting values, $\bar{q} \rightarrow \pm q_{\alpha\nu}^0$. This is consistent with the result of Ref. [5] that creation of one $c\nu \neq c0$ pseudoparticle at $\pm q_{c\nu}^0 = \pm[\pi - 2k_F]$ (and one $s\nu \neq s1$ pseudoparticle at $\pm q_{s\nu}^0 = \pm[k_{F\uparrow} - k_{F\downarrow}]$) leads to a change ν in the number of lattice sites doubly occupied by both electrons and rotated electrons (and singly occupied by both spin-down electrons and spin-down rotated electrons). The same result applies to the corresponding $c\nu \neq c0$ pseudofermion (and $s\nu \neq s1$ pseudofermion). Here we study the role of such a symmetry in the scattering properties of the $\alpha\nu \neq c0, s1$ pseudofermions for initial ground states with densities in the ranges $0 < n < 1$ and $0 < m < n$. (As mentioned above, the problem is addressed for initial ground states with densities $n = 1$ and/or $m = 0$ in Sec. IV-D.) First we use the phase-shift expressions involving $\alpha\nu \neq c0, s1$ pseudofermions of canonical momentum $\bar{q} = \pm q_{\alpha\nu}^0$ to find important information about their properties as scatterers and scattering centers. Next we use such properties to clarify the effects of their invariance under the electron - rotated-electron unitary transformation. Below in this subsection we use in general the notations $\alpha\nu$ and $\alpha'\nu'$ to denote the branch of the pseudofermion or pseudofermion-hole scatterers and scattering centers, respectively.

Our first goal is to show that the $\alpha\nu \neq c0, s1$ pseudofermions of limiting canonical momentum $\bar{q} = \pm q_{\alpha\nu}^0$ are not active scatterers. By active scatterers we mean scatterers whose overall phase shifts generated by the ground-state - excited-energy-eigenstate transitions lead to a shift of the corresponding canonical-momentum values. Let us confirm that the ground-state limiting canonical-momentum values $\bar{q} = \pm q_{\alpha\nu}^0$ of the pseudofermions belonging to $\alpha\nu \neq c0, s1$ branches are not shifted by the ground-state - excited-energy-eigenstate transitions. In contrast to the $c0$ pseudoparticles and the usual band particles and Fermi-liquid quasi-particles, the band bare-momentum limiting values $\pm q_{\alpha\nu}^0 = \pm\pi[N_{\alpha\nu}^* - 1]/L$ of the $\alpha\nu \neq c0, s1$ pseudoparticles can be changed by the above transitions. Since $\pm\Delta q_{\alpha\nu}^0 = \pm\pi\Delta N_{\alpha\nu}^*/L$, such an exotic behavior occurs when the deviation $\Delta N_{\alpha\nu}^*$, Eq. (18), generated by the ground-state - excited-energy-eigenstate transition is finite. From use of Eqs. (A8)-(A13) of Ref. [6] we find that the two-pseudofermion shifts $\pi\Phi_{\alpha\nu, \alpha'\nu'}(\iota q_{\alpha\nu}^0, q)$ which can be written as $\pi\Phi_{\alpha\nu, c\nu'}(\iota[\pi - 2k_F], q)$ and $\pi\Phi_{\alpha\nu, s\nu'}(\iota[k_{F\uparrow} - k_{F\downarrow}], q)$ for $\alpha\nu = c\nu \neq c0$ and $\alpha\nu = s\nu \neq s1$, respectively, have the following expression,

$$\pi\Phi_{\alpha\nu, \alpha'\nu'}(\iota q_{\alpha\nu}^0, q) = \frac{\iota\pi}{2} \left[\delta_{\alpha'\nu', c0}(\delta_{\alpha, c} - \delta_{\alpha, s}) + \delta_{\alpha, \alpha'}(-\delta_{\nu, \nu'} + \nu + \nu' - |\nu - \nu'|) \right]; \quad \iota = \pm 1; \quad \alpha\nu \neq c0, s1, \quad (49)$$

where $q \neq \iota q_{\alpha\nu}^0$ for $\alpha'\nu' = \alpha\nu \neq c0, s1$. Use of this two-pseudofermion phase-shift expression in the general overall scattering phase-shift expression (4) leads to,

$$\frac{Q_{\alpha\nu}^\Phi(\iota q_{\alpha\nu}^0)}{2} = -\frac{\iota\pi}{2} \left[\Delta N_{\alpha\nu} - [\delta_{\alpha, c} - \delta_{\alpha, s}]\Delta N_{c0} - \sum_{\nu'=1}^{\infty} (\nu + \nu' - |\nu - \nu'|)\Delta N_{\alpha\nu'} \right]; \quad \iota = \pm 1; \quad \alpha\nu \neq c0, s1. \quad (50)$$

Comparison of this expression with that of $\iota\Delta q_{\alpha\nu} = \iota\pi\Delta N_{\alpha\nu}^*/L$ where $N_{\alpha\nu}^* = N_{\alpha\nu} + N_{\alpha\nu}^h$ and $N_{\alpha\nu}^h$ is provided in Eq. (24) confirms that $Q_{\alpha\nu}^\Phi(\iota q_{\alpha\nu}^0)/L = \iota\Delta q_{\alpha\nu}$. The value of the two-pseudofermion phase shift (49) is not well defined for $q \neq \iota q_{\alpha\nu}^0$ and $\alpha'\nu' = \alpha\nu \neq c0, s1$. However, from the rapidity expression $\Lambda_{\alpha\nu}(q) = \Lambda_{\alpha\nu}^0(\bar{q}(q))$ for the PS excited energy-eigenstates, where $\Lambda_{\alpha\nu}^0(q)$ is the expression for the initial ground state [6], one confirms that the relation $Q_{\alpha\nu}^\Phi(\iota q_{\alpha\nu}^0)/L = \iota\Delta q_{\alpha\nu}$ is valid for all PS excited states. Indeed, that the rapidity functions of the excited energy states of a given initial ground state equal those of the latter state provided that in the argument of such functions the ground-state bare momentum is replaced by the excited-state canonical momentum implies that the corresponding bare-momentum and canonical-momentum bands have precisely the same width.

This result has a deep physical meaning: the scattering phase shift leads to a canonical-momentum shift $Q_{\alpha\nu}^{\Phi}(\pm q_{\alpha\nu}^0)/L$ that exactly cancels the bare-momentum shift $\pm\Delta q_{\alpha\nu}$. This implies that the overall canonical-momentum shift $\pm\Delta\bar{q}_{\alpha\nu} = \pm\Delta q_{\alpha\nu} + Q_{\alpha\nu}^{\Phi}(\pm q_{\alpha\nu}^0)/L$ indeed vanishes. It follows that for the $\alpha\nu \neq c0, s1$ pseudofermions the limiting canonical momenta have the same values, $\pm q_{\alpha\nu}^0$, both for the ground state and excited energy eigenstates and thus for $\bar{q} = \pm q_{\alpha\nu}^0$ such objects are not active scatterers.

Let us next investigate the properties of such pseudofermions as scattering centers. By use of Eqs. (A1)-(A13) of Ref. [6] and Eqs. (A.11)-(A.14) of Ref. [16] we find after some algebra that the two-pseudofermion shifts $\pi\Phi_{\alpha\nu, \alpha'\nu'}(q, \iota q_{\alpha'\nu'}^0)$ which can be written as $\pi\Phi_{\alpha\nu, c\nu'}(q, \iota[\pi - 2k_F])$ and $\pi\Phi_{\alpha\nu, s\nu'}(q, \iota[k_{F\uparrow} - k_{F\downarrow}])$ for $\alpha'\nu' = c\nu' \neq c0$ and $\alpha'\nu' = s\nu' \neq s1$, respectively, have the following general form,

$$\begin{aligned} \pi\Phi_{\alpha\nu, \alpha'\nu'}(q, \iota q_{\alpha'\nu'}^0) &= \frac{\iota\pi}{2} \left[\delta_{\alpha\nu, c0} - \delta_{\alpha, \alpha'}(\nu + \nu' - |\nu - \nu'|) \right] \\ &+ \frac{\iota}{2} \sum_{\nu'=\pm 1} \left[\pi\Phi_{\alpha\nu c0}(q, \iota' 2k_F) - \delta_{\alpha', s} 2\pi\Phi_{\alpha\nu s1}(q, \iota' k_{F\downarrow}) \right]; \quad \iota = \pm 1; \quad \alpha'\nu' \neq c0, s1. \end{aligned} \quad (51)$$

Here $\alpha\nu$ is any of the branches with finite pseudofermion occupancy for the excited state under consideration and the values of q are such that $|q| < q_{\alpha'\nu'}^0$ for $\alpha\nu = \alpha'\nu' \neq c0, s1$ and otherwise can have any value and thus correspond to *all* active $\alpha\nu$ scatterers of that state. The form of these exact two-pseudofermion phase-shift expressions reveals that, except for the constant phase-shift terms, creation of one $c\nu' \neq c0$ pseudofermion (and one $s\nu' \neq s1$ pseudofermion) at canonical momentum $\iota[\pi - 2k_F]$ (and $\iota[k_{F\uparrow} - k_{F\downarrow}]$) is felt by the $\alpha\nu$ pseudofermion or hole active scatterers as a shift $\iota\pi/L$ of both $c0$ bare-momentum *Fermi* points (and a shift $\iota\pi/L$ of both $c0$ bare-momentum *Fermi* points and a shift $-\iota 2\pi/L$ of both $s1$ bare-momentum *Fermi* points). Thus, such scatterers effectively feel that they are scattered by $c0$ *Fermi*-point current shifts (and $c0$ and $s1$ *Fermi*-point current shifts), rather than by the $c\nu' \neq c0$ (and $s\nu' \neq s1$) pseudofermion created at canonical momentum $\iota[\pi - 2k_F]$ (and $\iota[k_{F\uparrow} - k_{F\downarrow}]$).

Active scattering centers are those which contribute to the scattering phase shift (4). For instance, small-momentum and low-energy $c0$ and $s1$ pseudofermion particle-hole processes in the vicinity of the *Fermi* points, called elementary processes (C) in Ref. [15], do not generate active scattering centers. Indeed, within such processes the phase shifts generated by the pseudofermion particle excitations exactly cancel those produced by the pseudofermion hole excitations. The part of the bare-momentum distribution-function deviation generated by $\alpha'\nu'$ active scattering centers can be written as $\Delta N_{\alpha'\nu'}^{NF}(q') + \Delta N_{\alpha'\nu'}^F(q')$. Here $\Delta N_{\alpha'\nu'}^{NF}(q')$ is generated by the processes called elementary processes (A) in Ref. [15], which create and annihilate (and create) $\alpha'\nu' = c0, s1$ pseudofermions (and $\alpha'\nu' \neq c0, s1$ pseudofermions) away from the *Fermi* points (and away from the limiting values $\pm q_{\alpha'\nu'}^0$). In turn, $\Delta N_{\alpha'\nu'}^F(q')$ is generated by the processes called elementary processes (B) in the same reference, which create and annihilate (and create) $\alpha'\nu' = c0, s1$ pseudofermions (and $\alpha'\nu' \neq c0, s1$ pseudofermions) at the *Fermi* points (and at the limiting values $\pm q_{\alpha'\nu'}^0$). In this subsection we are mostly interested in the scattering centers associated with the deviation $\Delta N_{\alpha'\nu'}^F(q')$, whose general expression reads,

$$\begin{aligned} \Delta N_{\alpha'\nu'}^F(q') &= \sum_{\iota=\pm 1} \left[\frac{\Delta N_{\alpha'\nu'}^F}{2} + \iota \Delta J_{\alpha'\nu'}^F \right] \delta_{q', \iota q_{\alpha'\nu'}^0}; \quad \alpha'\nu' = c0, s1; \\ &= \sum_{\iota=\pm 1} \left[\frac{N_{\alpha'\nu'}^F}{2} + \iota J_{\alpha'\nu'}^F \right] \delta_{q', \iota q_{\alpha'\nu'}^0}; \quad \alpha'\nu' \neq c0, s1. \end{aligned} \quad (52)$$

Here the deviation numbers (and numbers) $\Delta N_{\alpha'\nu'}^F$ (and $N_{\alpha'\nu'}^F$) are such that $\Delta N_{\alpha'\nu'} = \Delta N_{\alpha'K'}^F + \Delta N_{\alpha'K'}^{NF}$ (and $N_{\alpha'\nu'} = N_{\alpha'K'}^F + N_{\alpha'K'}^{NF}$). They can be expressed as $\Delta N_{\alpha'\nu'}^F = \Delta N_{\alpha'\nu', +1}^F + \Delta N_{\alpha'\nu', -1}^F$ (and $N_{\alpha'\nu'}^F = N_{\alpha'\nu', +1}^F + N_{\alpha'\nu', -1}^F$), where $\Delta N_{\alpha'\nu', \pm 1}^F$ is the deviation in the number of $\alpha'\nu'$ pseudofermions at the right (+1) and left right (-1) *Fermi* point (and $N_{\alpha'\nu', \iota}^F$ is the number of $\alpha'\nu'$ pseudofermions created at $\iota q_{\alpha'\nu'}^0$ with $\iota = \pm 1$). The associated deviation current numbers (and current numbers) read $2\Delta J_{\alpha'\nu'}^F = \Delta N_{\alpha'\nu', +1}^F - \Delta N_{\alpha'\nu', -1}^F$ (and $2J_{\alpha'\nu'}^F = N_{\alpha'\nu', +1}^F - N_{\alpha'\nu', -1}^F$).

According to the PDT of Refs. [14, 15], the elementary processes (A), (B), and (C) mentioned above lead to qualitatively different contributions to the spectral-weight distributions. The PDT studies of these references considered that creation of $c\nu' \neq c0$ and $s\nu' \neq s1$ pseudofermions at the limiting bare-momentum values is felt by both the $c0$ and $s1$ scatterers as effective $c0$ scattering centers and effective $c0$ and $s1$ scattering centers, respectively. However, in these studies that was only considered for $c0$ and $s1$ scatterers of bare-momentum value $q = \pm q_{F\alpha\nu}^0$. We emphasize that the general two-pseudofermion expression (51) generalizes that result to *all* active $\alpha\nu$ scatterers, including both $\alpha\nu = c0, s1$ and $\alpha\nu \neq c0, s1$ active scatterers of arbitrary bare momentum q .

From the linearity in the deviations of the overall scattering phase shift (4) one can write $Q_{\alpha\nu}^{\Phi}(q)/2 = Q_{\alpha\nu}^{\Phi(NF)}(q)/2 + Q_{\alpha\nu}^{\Phi(F)}(q)/2$ where $Q_{\alpha\nu}^{\Phi(NF)}(q)/2$ and $Q_{\alpha\nu}^{\Phi(F)}(q)/2$ result from the $\alpha'\nu'$ scattering centers associated with the deviations $\Delta N_{\alpha'\nu'}^{NF}(q')$ and $\Delta N_{\alpha'\nu'}^F(q')$, respectively. Also the part of the total momentum deviation (20) associated with the

elementary processes (A) and (B) can be written as $\Delta P^{NF} + \Delta P^F$. After some algebra involving the use of Eqs. (4), (20), (51), and (52) we reach the following expressions for these quantities,

$$\frac{Q_{\alpha\nu}^{\Phi(NF)}(q)}{2} = \sum_{\alpha'\nu'} \sum_{q'} \pi \Phi_{\alpha\nu, \alpha'\nu'}(q, q') \Delta N_{\alpha'\nu'}^{NF}(q'), \quad (53)$$

$$\begin{aligned} \frac{Q_{\alpha\nu}^{\Phi(F)}(q)}{2} = & \sum_{\alpha'\nu'=c0, s1} \sum_{\iota'=\pm 1} \pi \Phi_{\alpha\nu, \alpha'\nu'}(q, \iota' q_{F\alpha'\nu'}^0) \frac{\Delta N_{\alpha'\nu'}^F}{2} \\ & + \sum_{\iota'=\pm 1} \iota' \pi \Phi_{\alpha\nu, c0}(q, \iota' 2k_F) \left[\Delta J_{c0}^F + \sum_{\nu'=1}^{\infty} J_{c\nu'}^F + \sum_{\nu'=2}^{\infty} J_{s\nu'}^F \right] \\ & + \sum_{\iota'=\pm 1} \iota' \pi \Phi_{\alpha\nu, s1}(q, \iota' k_{F\downarrow}) \left[\Delta J_{s1}^F - 2 \sum_{\nu'=2}^{\infty} J_{s\nu'}^F \right] \\ & + \sum_{\alpha'\nu' \neq c0, s1} \pi \left[\delta_{\alpha\nu, c0} - \delta_{\alpha, \alpha'} (\nu + \nu' - |\nu - \nu'|) \right] J_{\alpha'\nu'}^F, \end{aligned} \quad (54)$$

and

$$\begin{aligned} \Delta P^{NF} = & \sum_{\alpha'\nu'=c0, s\nu'} \sum_{q'} q' \Delta N_{\alpha'\nu'}^{NF}(q') + \sum_{c\nu' \neq c0} \sum_{q'} [\pi - q'] \Delta N_{c\nu'}^{NF}(q'); \\ \Delta P^F = & \pi [L_{c, -1/2} + \sum_{\nu'=1}^{\infty} \nu' N_{c\nu'}^F] + 4k_F \left[\Delta J_{c0}^F + \sum_{\nu'=1}^{\infty} J_{c\nu'}^F + \sum_{\nu'=2}^{\infty} J_{s\nu'}^F \right] + 2k_{F\downarrow} \left[\Delta J_{s1}^F - 2 \sum_{\nu'=2}^{\infty} J_{s\nu'}^F \right]. \end{aligned} \quad (55)$$

The general expression of the phase shift $Q_{\alpha\nu}^{\Phi(F)}(q)/2$ given in Eq. (54) is valid for all active $\alpha\nu$ scatterers which for the $\alpha\nu \neq c0, s1$ branches refer to bare-momentum values in the range $|q| < q_{\alpha\nu}^0$. In the ΔP^F expression of Eq. (55) we have included the contribution from the $-1/2$ Yang holons. (The momentum contributions from the $+1/2$ Yang holons and $\pm 1/2$ HL spinons vanish.) Note that the current contributions to the momentum spectrum ΔP^F given in Eq. (55), which multiply twice the value of the $c0$ and $s1$ *Fermi momenta* $2k_F$ and $k_{F\downarrow}$ are identical to the current contributions to the scattering phase shift (54) which multiply the phase shifts $\pi \Phi_{\alpha\nu, c0}(q, \iota' 2k_F)$ and $\pi \Phi_{\alpha\nu, s1}(q, \iota' k_{F\downarrow})$, respectively.

That the $c\nu' \neq c0$ pseudofermions and $s\nu' \neq s1$ pseudofermions created at limiting canonical-momentum values are not active scatterers and are felt by the active scatterers as effective $c0$ scattering centers and $c0$ and $s1$ scattering centers, respectively, follows from their invariance under the electron - rotated-electron unitary transformation. Such properties reflect the following important decoupling: as $\bar{q} \rightarrow \pm q_{\alpha'\nu'}^0$, the $c\nu' \neq c0$ pseudofermion (and $s\nu' \neq s1$ pseudofermion) separates into $2\nu'$ independent holons (and $2\nu'$ independent spinons) and a $c\nu'$ (and $s\nu'$) *FP* current scattering center. By independent holons and spinons we mean those which remain invariant under the electron - rotated-electron unitary transformation. It follows that the Yang holons and HL spinons are also independent holons and spinons, respectively. By a $c\nu'$ (and $s\nu'$) *FP* current scattering center we mean the elementary current $J_{c\nu'}^F = \iota/2$ (and $J_{s\nu'}^F = \iota/2$) generated by creation of one $c\nu' \neq c0$ pseudofermion (and one $s\nu' \neq s1$ pseudofermion) at $\bar{q} = \iota[\pi - 2k_F]$ (and $\bar{q} = \iota[k_{F\uparrow} - k_{F\downarrow}]$). As confirmed by the form of the phase shifts given in Eqs. (51) and (54), such elementary currents are felt by the $\alpha\nu$ active scatterers as elementary shifts of both $c0$ *Fermi*-points (and both $c0$ and both $s1$ *Fermi*-points). This justifies the designation *FP* (from *Fermi*-points) current scattering center.

The $c\nu' \neq c0$ and $s\nu' \neq s1$ pseudofermions of limiting canonical-momentum value $\pm q_{\alpha'\nu'}^0$ have energy $2\nu'\mu$ and $2\nu'\mu_0 H$, respectively, relative to the ground state. The energy of one $-1/2$ independent holon and one $+1/2$ independent holon (and one $-1/2$ independent spinon and one $+1/2$ independent spinon) relative to that state is 2μ and zero (and $2\mu_0 H$ and zero), respectively. Thus, for $\bar{q} = \pm q_{\alpha'\nu'}^0$ the $c\nu'$ pseudofermion (and $s\nu'$ pseudofermion) energy is additive in those of the corresponding $\nu' - 1/2$ holons and $\nu' + 1/2$ holons (and $\nu' - 1/2$ spinons and $\nu' + 1/2$ spinons). This is consistent with the $\alpha'\nu' \neq c0, s1$ pseudofermions loosing their composite character as $\bar{q} \rightarrow \pm q_{\alpha'\nu'}^0$. While the whole pseudofermion energy goes over to the $\nu' - 1/2$ holons or $\nu' - 1/2$ spinons, part or the whole pseudofermion momentum, respectively, is transferred over to the *FP* current scattering center. In contrast, for $|\bar{q}| < q_{\alpha'\nu'}^0$ the $c\nu'$ pseudofermion (and $s\nu'$ pseudofermion) energy relative to the ground state reads $\epsilon_{c\nu'}(q) = 2\nu'\mu + \epsilon_{c\nu'}^0(q)$ (and $\epsilon_{s\nu'}(q) = 2\nu'\mu_0 H + \epsilon_{s\nu'}^0(q)$), where the energy $\epsilon_{\alpha'\nu'}^0(q) \neq 0$ is defined in Eqs. (C.17), (C.18), (C.20), and (C.21) of Ref. [4]. That the energy $\epsilon_{\alpha'\nu'}^0(q)$ is finite is consistent with the composite character of the $2\nu'$ holons and $2\nu'$ spinons of the $c\nu'$ and $s\nu'$ pseudofermion, respectively. In this general case the pseudofermions are not invariant under

the electron - rotated-electron unitary transformation. The holon or spinon degrees of freedom are then combined with the pseudofermion scattering part and, therefore, the $c\nu' \neq c0$ and $s\nu' \neq s1$ pseudofermions created under a ground-state - excited-state transition are felt by the $\alpha\nu$ pseudofermion and $\alpha\nu$ pseudofermion-hole active scatterers as independent scattering centers, unrelated to the $c0$ and $s1$ *Fermi* points.

In what the independent holons associated with a $c\nu'$ pseudofermion of canonical momentum $\bar{q} = \pm q_{c\nu'}^0 = \pm[\pi - 2k_F]$ is concerned, the main difference between having (i) $L_{c,-1/2} = \nu' - 1/2$ Yang holons plus $L_{c,+1/2} = \nu' + 1/2$ Yang holons and (ii) one $c\nu'$ pseudofermion is that the $2\nu'$ Yang holons have a total η -spin value $S_c = \nu'$ whereas for the $2\nu'$ holons associated with the pseudofermion the total η -spin value is $S_c = 0$ and thus corresponds to a η -spin singlet configuration. (The same holds for the $s\nu'$ pseudofermions provided that one replaces holons by spinons and Yang holons by HL spinons.) As the Yang holons and HL spinons, the $2\nu'$ independent holons ($\alpha' = c$) or $2\nu'$ independent spinons ($\alpha' = s$) of a $\alpha'\nu'$ pseudofermion of canonical momentum $\bar{q} = \pm q_{\alpha'\nu'}^0$ are neither scatterers nor scattering centers.

Within the PDT of Refs. [14, 15], the $\alpha\nu = c0, s1$ canonical-momentum *Fermi*-point deviations $\iota \Delta \bar{q}_{F\alpha\nu}$ play a central role in the spectral-function expressions through the related quantity $2\Delta_{\alpha\nu}^\iota = [\iota \Delta \bar{q}_{F\alpha\nu} / (2\pi/L)]^2$ where $\iota \Delta \bar{q}_{F\alpha\nu} / (2\pi/L) = \iota \Delta N_{\alpha\nu, \iota}^F + Q_{\alpha\nu}^\Phi(\iota q_{F\alpha\nu}^0) / 2\pi$. We thus emphasize that the validity of the corresponding expressions (37) and (40) of Ref. [15] follows from the general expression for the phase shift $Q_{\alpha\nu}^{\Phi(F)}(q)/2$ given in Eq. (54).

D. INVARIANCE UNDER BOTH THE ELECTRON - ROTATED-ELECTRON AND PSEUDOPARTICLE - PSEUDOFORMION TRANSFORMATIONS FOR $n = 1$ AND $m = 0$ DENSITIES

The general pseudofermion scattering theory also applies to initial ground states with densities $n = 1$ and/or $m = 0$, provided that the specific features reported here are taken into account. For an initial ground state with electronic density $n = 1$ (and spin density $m = 0$) one has that $N_{c\nu}^* = 0$ (and $N_{s\nu}^* = 0$) for the $c\nu$ (and $s\nu$) band and thus the corresponding pseudofermion branch does not exist. For simplicity, we focus our attention onto excited energy eigenstates of such an initial ground state with a single $c\nu \neq c0$ pseudofermion (and a single $s\nu \neq s1$ pseudofermion). For these excited states, $N_{c\nu}^* = 1$ (and $N_{s\nu}^* = 1$) and the corresponding $c\nu$ (and $s\nu$) bare-momentum band reduces to the bare momentum zero.

A property specific to $n = 1$ (and $m = 0$) initial ground states is that a transition to an excited state involving creation of one $c\nu \neq c0$ pseudofermion (and one $s\nu \neq s1$ pseudofermion) always also involves creation of 2ν $c0$ pseudofermion holes (and $2(\nu - 1)$ $s1$ pseudofermion holes). Furthermore, if one considers the $n = 1$ and $m = 0$ initial ground state it follows from Eqs. (B9), (B21), (B22), and (B23) of Appendix B that $\Phi_{c0, s\nu'}(q, 0) = \Phi_{c\nu, s\nu'}(q, 0) = \Phi_{s\nu, c\nu'}(q, 0) = 0$ for $\nu \geq 1$ and $\nu' \geq 1$. In turn, we find below that the value of the phase shifts $\pi \Phi_{c0, c\nu}(q, 0)$ (and $\pi \Phi_{s1, s\nu}(q, 0)$) is fully determined by the 2ν (and $2(\nu - 1)$) bare-momentum values of the $c0$ pseudofermion-hole (and $2(\nu - 1)$ $s1$ pseudofermion-hole) scattering centers. Thus, for excited states of the $n = 1$ and $m = 0$ initial ground state the $c0$ and $s1$ scatterers feel the created $c\nu \neq c0$ pseudofermion (and $s\nu \neq s1$ pseudofermion) as $c0$ effective scattering centers (and $s1$ effective scattering centers).

These effective scattering centers are different from those considered above for the excited states of ground states with electronic density (and spin density) in the range $0 < n < 1$ (and $0 < m < n$). Indeed, for the excited states of a $n = 1$ (and $m = 0$) initial ground state considered here the current number $J_{c\nu}^F$ (and $J_{s\nu}^F$) vanishes and thus there are no $c\nu \neq c0$ (and $s\nu \neq s1$) *FP* current scattering centers. The occurrence of the type of $c0$ (and $s1$) effective scattering centers considered in this section follows from the non-scatterer character of the corresponding $c\nu \neq c0$ pseudofermions (and $s\nu \neq s1$ pseudofermion), as discussed below.

In spite of the lack of ground-state $c\nu \neq c0$ (and $s\nu \neq s1$) pseudofermion bands, the scattering theory can be generalized to an initial ground state with electronic density $n = 1$ (and $m = 0$): we recall that the “in” asymptote one-pseudofermion scattering states do not contribute to the direct-product expression of the initial ground state but rather to that of the “in” state defined in Sec. III-C. For the excited states of $n = 1$ (and $m = 0$) ground states considered here, the $c\nu \neq c0$ (and $s\nu \neq s1$) bare-momentum band corresponds to a single value at $q = 0$. For initial ground states with densities $0 < n < 1$ and $0 < m < n$ the scattering canonical-momentum shift $Q_{\alpha\nu}^\Phi(q)/L$, Eq. (4), has the same value whether one uses the ground-state rapidity functions $\Lambda_{\alpha\nu}^0(q)$ and $\Lambda_{\alpha'\nu'}^0(q')$ or the corresponding “out”-state (excited-energy-eigenstate) rapidity functions $\Lambda_{\alpha\nu}(q)$ and $\Lambda_{\alpha'\nu'}(q')$ in the general expression (6) for the two-pseudofermion phase shifts. Indeed, these two alternative definitions of the two-pseudofermion phase shifts lead to the same value for the functional $Q_{\alpha\nu}^\Phi(q)/L$ up to contributions of order $1/L$. In turn, the general pseudofermion scattering theory introduced above also applies to initial ground states with densities $n = 1$ and/or $m = 0$ provided that the following procedure is performed:

– Since for a $n = 1$ and/or $m = 0$ initial ground state there are no $c\nu \neq c0$ and/or $s\nu \neq s1$ pseudofermion bands, the two-pseudofermion expression (6) should be replaced by $\Phi_{\alpha\nu, \alpha'\nu'}(q, q') = \bar{\Phi}_{\alpha\nu, \alpha'\nu'}(4t \Lambda_{\alpha\nu}(q)/U, 4t \Lambda_{\alpha'\nu'}(q')/U)$,

where the rapidity function $\Lambda_{\alpha\nu}(q)$ (and $\Lambda_{\alpha'\nu'}(q')$) is that of the excited state and $\alpha\nu = c\nu$ with $\nu > 0$ and/or $\alpha\nu = s\nu$ with $\nu > 1$ (and $\alpha'\nu' = c\nu'$ with $\nu' > 0$ and/or $\alpha'\nu' = s\nu'$ with $\nu' > 1$), and otherwise is that of the initial ground state. Since the former rapidity functions are those of the excited state under consideration, it follows that for the particular case of such an initial ground state the quantity (6) is a functional rather than a function.

We found in the previous subsection that for initial ground states with densities in the ranges $0 < n < 1$ and $0 < m < n$, the $c0$, $s1$ pseudofermions with limiting canonical momentum given by $\pm q_{\alpha\nu}^0$ are not active scatterers. As discussed below, for the excited energy eigenstates considered in this subsection the $c\nu \neq c0$ (and $s\nu \neq s1$) band reduces to a single discrete canonical momentum value at $\bar{q} = 0$. Such excited states have electronic density $n \rightarrow 1$ (and spin density $m \rightarrow 0$) and thus the single $\bar{q} = 0$ value corresponds to the limiting canonical momentum values $\pm q_{c\nu}^0 = \pm[\pi - 2k_F]$ (and $\pm q_{s\nu}^0 = \pm[k_{F\uparrow} - k_{F\downarrow}]$) such that $\pm q_{c\nu}^0 \rightarrow 0$ as $n \rightarrow 1$ (and $\pm q_{s\nu}^0 \rightarrow 0$ as $m \rightarrow 0$). It follows that similarly to the pseudofermions considered in the previous subsection, the $c\nu \neq c0$ pseudofermion (and $s\nu \neq s1$ pseudofermion) of the excited states considered here are invariant under the electron - rotated-electron unitary transformation. Indeed, it follows from the results of Ref. [5] that creation of such a $c\nu \neq c0$ pseudofermion (and $s\nu \neq s1$ pseudofermion) leads to a change ν in the number of lattice sites doubly occupied by both electrons and rotated electrons (and singly occupied by both spin-down electrons and spin-down rotated electrons). As a result, they separate into 2ν independent holons (and spinons). Such 2ν independent holons (and 2ν independent spinons) are fully decoupled from the above mentioned $c0$ effective scattering centers (and $s1$ effective scattering centers) also associated with the $c\nu \neq c0$ pseudofermion (and $s\nu \neq s1$ pseudofermion). It is then required that they are not active scatterers. However, for the class of excited states considered here a necessary condition for such objects not being active scatterers is that they are not scatterers at all. Indeed, once the $c\nu \neq c0$ (and $s\nu \neq s1$) bare-momentum band of the “in”

state has a single value at $q = 0$, it is required that the corresponding canonical-momentum band of the “out” state has also a single value at $\bar{q} = 0$. This implies both that $Q_{\alpha\nu}^\Phi(0)/L = 0$ and $Q_{\alpha\nu}(0)/L = 0$, and thus that the $c\nu \neq c0$ (and $s\nu \neq s1$) pseudoparticle remains invariant under the pseudoparticle - pseudofermion unitary transformation. Indeed, since the $c\nu \neq c0$ (and $s\nu \neq s1$) band does not exist for the initial ground state, one has that the scatterless overall phase shift vanishes, $Q_{\alpha\nu}^0/2 = 0$, and thus for the above reasoning the overall phase shift is such that $Q_{\alpha\nu}(0)/2 = Q_{\alpha\nu}^\Phi(0)/2 = 0$. It follows that the corresponding $c\nu \neq c0$ (and $s\nu \neq s1$) pseudofermion is not a scatterer.

However, the requirement that such a $c\nu \neq c0$ (and $s\nu \neq s1$) pseudofermion is not a scatterer and thus that $Q_{c\nu}^\Phi(0)/L = 0$ (and $Q_{s\nu}^\Phi(0)/L = 0$) imposes a specific form to the corresponding two-pseudofermion phase shifts $\Phi_{c\nu, \alpha'\nu'}(0, q')$ (and $\Phi_{s\nu, \alpha'\nu'}(0, q')$). Since such objects are neither scatterers nor scattering centers, the quantities $\pi \Phi_{c\nu, \alpha'\nu'}(0, q')$ (and $\pi \Phi_{s\nu, \alpha'\nu'}(0, q')$) are not real two-pseudofermion phase shifts: they are just effective two-pseudofermion phase shifts whose values are such that the overall scattering phase shift $Q_{c\nu}^\Phi(0)/2$ (and $Q_{s\nu}^\Phi(0)/2$) vanishes. Thus, there is no requirement that they do obey Levinson's Theorem. Also the phase shifts $\pi \Phi_{\alpha'\nu', c\nu}(q', 0)$ and $\pi \Phi_{\alpha'\nu', s\nu}(q', 0)$ are not required to obey that theorem for $\alpha'\nu' = c0, s1$, once they refer to effective $c0$ and $s1$ scattering centers, as confirmed below.

Let us consider three types of excited energy eigenstates of the above class. Those are states with finite pseudofermion occupancy for the $c0$ and $s1$ bands plus (a) one $c\nu \neq c0$ pseudofermion and one $s\nu' \neq s1$ pseudofermion, (b) one $c\nu \neq c0$ pseudofermion, and (c) one $s\nu' \neq s1$ pseudofermion. For simplicity, we consider excited energy eigenstates of the $n = 1$ and $m = 0$ initial ground state. Such a ground state is described by full $c0$ and $s1$ pseudofermion bands whose *Fermi* bare momentum reads $2k_F = \pi$ and $k_{F\downarrow} = k_{F\uparrow} = k_F = \pi/2$, respectively. Thus, from the use of Eq. (B.11) of Ref. [4] we have that $\Delta N_{c0} = -\Delta N_{c0}^h = -2\nu$, $\Delta N_{s1} = -\nu - \nu'$, $\Delta N_{s1}^h = 2(\nu' - 1)$ for the excited states (a), $\Delta N_{c0} = -\Delta N_{c0}^h = -2\nu$, $\Delta N_{s1} = -\nu$, and $\Delta N_{s1}^h = 0$ for the excited states (b), and $\Delta N_{c0} = -\Delta N_{c0}^h = 0$, $\Delta N_{s1} = -\nu'$, and $\Delta N_{s1}^h = 2(\nu' - 1)$ for the excited states (c). On the other hand, according to Eqs. (B9), (B10), and (B23) of Appendix B, for $m \rightarrow 0$ and $n \rightarrow 1$ the two-pseudofermion phase shifts that contribute to $Q_{c\nu}^\Phi(0)/2$ and $Q_{s\nu}^\Phi(0)/2$ simplify to $\bar{\Phi}_{c\nu, s\nu'}(r, r') = \bar{\Phi}_{s\nu', c0}(r', r) = \bar{\Phi}_{s\nu', c\nu}(r', r) = 0$, $\bar{\Phi}_{c\nu, c0}(r, r') = \frac{1}{\pi} \arctan\left(\frac{r-r'}{\nu}\right)$, and $\bar{\Phi}_{s\nu', s1}(r', r) = \frac{1}{\pi} \arctan\left(\frac{r'-r}{\nu'-1}\right)$ for $\nu' > 1$ and $\nu > 0$. It follows that for the excited energy eigenstates (a)-(c) the equation $Q_{c\nu}^\Phi(0)/2 = 0$ and/or $Q_{s\nu}^\Phi(0)/2 = 0$ leads to,

$$\begin{aligned} \sum_{l=1}^{2\nu} \arctan\left(\frac{4t}{\nu U} \left[\Lambda_{c\nu}(0) - \Lambda_{c0}^0(q_l)\right]\right) &= 0, \quad \nu > 0, \\ \sum_{l=1}^{2(\nu'-1)} \arctan\left(\frac{4t}{(\nu'-1)U} \left[\Lambda_{s\nu'}(0) - \Lambda_{s1}^0(q'_l)\right]\right) &= 0, \quad \nu' > 1. \end{aligned} \quad (56)$$

Here the first and second equations refer to the $c0$ branch and both the states (a) and (b) and to the $s1$ branch and both the states (a) and (c), respectively. In these equations the set of 2ν values $\{q_l\}$ and of $2(\nu' - 1)$ values

$\{q'_l\}$ correspond to the excited-energy-eigenstate $c0$ pseudofermion holes and $s1$ pseudofermion holes, respectively. Moreover, the ground-state rapidity functions $\Lambda_{c0}^0(q) \equiv \sin k_0(q)$ and $\Lambda_{s1}^0(q)$ can be defined in terms of their inverse functions given in Eq. (A.1) of Ref. [15]. For $n = 1$ and $m = 0$ the expressions provided in the latter equation lead to,

$$q = k_0(q) + 2 \int_0^\infty d\omega \frac{\sin(\omega \sin k_0(q))}{\omega (1 + e^{2\omega U/4t})} J_0(\omega); \quad q = \int_0^\infty d\omega \frac{\sin(\omega \Lambda_{s1}^0(q))}{\omega \cosh(\omega U/4t)} J_0(\omega), \quad (57)$$

where $J_0(\omega)$ is a Bessel function. These equations define the inverse of the functions $k_0(q)$ and $\Lambda_{s1}^0(q)$, respectively. (To arrive to the second equality of Eq. (56) we used that $\Lambda_{s1}^0(\pm k_F) = \Lambda_{s1}^0(\pm \pi/2) = \pm \infty$.) The form of the equalities given in Eq. (56) reveals that the corresponding solutions $\Lambda_{c\nu}(0) = \Lambda_{c\nu}(0, \{q_l\})$ and/or $\Lambda_{s\nu'}(0) = \Lambda_{s\nu'}(0, \{q'_l\})$ are functions of the above sets of bare-momentum values $\{q_l\}$ and $\{q'_l\}$, respectively.

We emphasize that the solution of the BA equations (13)-(16) of Ref. [4] for the above excited states leads to functions $\Lambda_{c\nu}(0) = \Lambda_{c\nu}(0, \{q_l\})$ and $\Lambda_{s\nu'}(0) = \Lambda_{s\nu'}(0, \{q'_l\})$ for the rapidities $\Lambda_{c\nu}(0)$ and $\Lambda_{s\nu'}(0)$, respectively, which also obey Eq. (56). This confirms that for such excited states the exact solution of the BA equations is indeed equivalent to imposing the symmetry requirement $Q_{c\nu}^\Phi(0)/2 = 0$ and $Q_{s\nu'}^\Phi(0)/2 = 0$ associated with the non-scatterer character of the corresponding $c\nu$ and $s\nu'$ pseudofermion, respectively.

The above functions $\Lambda_{c\nu}(0) = \Lambda_{c\nu}(0, \{q_l\})$ and $\Lambda_{s\nu'}(0) = \Lambda_{s\nu'}(0, \{q'_l\})$ are to be used in the following expressions,

$$\begin{aligned} \pi \Phi_{c\nu, c0}(0, q') &= \pi \bar{\Phi}_{c\nu, c0} \left(\frac{4t \Lambda_{c\nu}(0, \{q_l\})}{U}, \frac{4t \Lambda_{c0}^0(q')}{U} \right) = \arctan \left(\frac{4t [\Lambda_{c\nu}(0, \{q_l\}) - \Lambda_{c0}^0(q')]}{\nu U} \right); \\ \pi \Phi_{s\nu', s1}(0, q') &= \pi \bar{\Phi}_{s\nu', s1} \left(\frac{4t \Lambda_{s\nu'}(0, \{q'_l\})}{U}, \frac{4t \Lambda_{s1}^0(q')}{U} \right) = \arctan \left(\frac{4t [\Lambda_{s\nu'}(0, \{q'_l\}) - \Lambda_{s1}^0(q')]}{(\nu' - 1) U} \right), \end{aligned} \quad (58)$$

so that $Q_{c\nu}^\Phi(0)/2 = 0$ and/or $Q_{s\nu'}^\Phi(0)/2 = 0$. The simplest case corresponds to $\nu = 1$ and/or $\nu' = 2$ such that solution of Eq. (56) leads to $\Lambda_{c1}(0, q_1, q_2) = [\Lambda_{c0}^0(q_1) + \Lambda_{c0}^0(q_2)]/2$ and/or $\Lambda_{s2}(0, q'_1, q'_2) = [\Lambda_{s1}^0(q'_1) + \Lambda_{s1}^0(q'_2)]/2$, respectively.

The requirement that the $c\nu \neq c0$ pseudofermion (and $s\nu \neq s1$ pseudofermion) considered here is not a scatterer implies that the corresponding rapidity function $\Lambda_{c\nu}(0) = \Lambda_{c\nu}(0, \{q_l\})$ (and $\Lambda_{s\nu'}(0) = \Lambda_{s\nu'}(0, \{q'_l\})$) does not in general vanishes and is the unique solution of the first (and second) equality of Eq. (56). Combination of this result with the two-pseudofermion phase shift of Eqs. (B21) and (B22) of Appendix B reveals that the $c0$ scatterer two-pseudofermion phase shift $\pi \Phi_{c0, c\nu}(q, 0)$ and the $s1$ scatterer two-pseudofermion phase shift $\pi \Phi_{s1, s\nu'}(q, 0)$ are such that,

$$\begin{aligned} \pi \Phi_{c0, c\nu}(q, 0) &= \pi \bar{\Phi}_{c0, c\nu} \left(\frac{4t \Lambda_{c0}^0(q)}{U}, \frac{4t \Lambda_{c\nu}(0, \{q_l\})}{U} \right) = -\arctan \left(\frac{4t [\Lambda_{c0}^0(q) - \Lambda_{c\nu}(0, \{q_l\})]}{\nu U} \right); \\ \pi \Phi_{s1, s\nu'}(q, 0) &= \pi \bar{\Phi}_{s1, s\nu'} \left(\frac{4t \Lambda_{s1}^0(q)}{U}, \frac{4t \Lambda_{s\nu'}(0, \{q'_l\})}{U} \right) = \arctan \left(\frac{4t [\Lambda_{s1}^0(q) - \Lambda_{s\nu'}(0, \{q'_l\})]}{(\nu' - 1) U} \right); \quad q \neq \pm k_F \\ &= \pm \frac{\pi}{\sqrt{2}}; \quad q = \pm k_F. \end{aligned} \quad (59)$$

We note that in addition to the $c0$ or $s1$ scatterer bare-momentum q , the two-pseudofermion phase shifts provided in Eq. (59) are functions of the set of 2ν bare-momentum values $\{q_l\}$ or $2(\nu' - 1)$ bare-momentum values $\{q'_l\}$ of the 2ν $c0$ pseudofermion-hole scattering centers or $2(\nu' - 1)$ $s1$ pseudofermion-hole scattering centers, respectively, also created under the ground-state - excited-energy-eigenstate transition. As a result of the creation of the $c\nu \neq c0$ (and $s\nu' \neq s1$) pseudofermion, the $c0$ (and $s1$) scatterers acquire the phase shift $\pi \Phi_{c0, c\nu}(q, 0)$ (and $\pi \Phi_{s1, s\nu'}(q, 0)$) whose value is fully controlled by the 2ν (and $2(\nu' - 1)$) bare-momentum values of the 2ν (and $2(\nu' - 1)$) $c0$ (and $s1$) pseudofermion-hole scattering centers. Thus, through the $\{q_l\}$ (and $\{q'_l\}$) momentum dependence of the phase shift $\pi \Phi_{c0, c\nu}(q, 0)$ (and $\pi \Phi_{s1, s\nu'}(q, 0)$), the $c0$ (and $s1$) scatterers feel the created $c\nu \neq c0$ (and $s\nu' \neq s1$) pseudofermion as 2ν $c0$ effective scattering centers (and $2(\nu' - 1)$ $s1$ effective scattering centers).

Similar results are obtained for excited states of initial ground states of density $n = 1$ and/or $m = 0$ with finite occupancy for a larger finite number of $\alpha\nu$ pseudofermions belonging to several $\alpha\nu \neq c0, s1$ branches, except that the number of equations defining the the set of rapidities $\{\Lambda_{\alpha\nu}\}$ is in general larger than above and each of these equations is more involved than the two equations given in Eq. (56). Importantly, such $\alpha\nu$ pseudofermions are also invariant under the electron - rotated-electron unitary transformation.

V. CONCLUDING REMARKS

In this paper we have shown that the non-perturbative and strongly correlated scattering problem described in terms of electrons by the Hamiltonian (1) considerably simplifies in terms of the pseudofermions of Refs. [4, 5, 6, 14, 15, 16]:

We have confirmed here that in terms of pseudofermion scattering the spectral and dynamical properties are controlled at all energy scales and for all values of the on-site electronic repulsion by two-pseudofermion zero-momentum forward scattering only. This agrees with the preliminary analysis of the problem of Ref. [29]. The matrix elements between the ground state and excited energy eigenstates of the corresponding spectral functions can be expressed in terms of the pseudofermion anticommutators [14, 15, 16]. Our results show that such anticommutators are controlled by two-pseudofermion zero-momentum forward scattering through the associated pseudofermion and hole S matrices.

Our results have also clarified the relation between the pseudofermion scattering properties and symmetry. Specifically, it was found for the metallic phase at finite spin density that the invariance under the electron - rotated-electron unitary transformation of the $\alpha\nu \neq c0$, $s1$ pseudofermions with limiting canonical momentum $\bar{q} = \pm q_{\alpha\nu}^0$, implies that such composite objects separate into 2ν independent holons ($\alpha = c$) or 2ν independent spinons ($\alpha = s$) and a current excitation. The latter excitation is felt by all $\alpha\nu$ active scatterers of arbitrary momentum as a shift of both $c0$ *Fermi* points and shifts of both $c0$ and $s1$ *Fermi* points, respectively. The effects of the invariance under both the electron - rotated-electron unitary transformation and pseudoparticle - pseudofermion unitary transformation were also studied for the Mott-Hubbard insulator at zero spin density. The property that only the holons and spinons that remain invariant under such a transformation are allowed to exist as independent quantum objects which are not part of 2ν -holon and 2ν -spinon composite pseudofermions, respectively, is general and also applies to the Yang holons and HL spinons. Such objects are neither scatterers nor scattering centers.

That our choice of scatterers and scattering centers profits from the transformation laws under the electron - rotated-electron unitary transformation justifies that all “in” and “out” states of the theory are excited energy eigenstates. All these states can be written as a direct product of “in” asymptote and “out” asymptote one-pseudofermion scattering states, respectively. Such a property combined with the simple form obtained for the pseudofermion and hole S matrices is behind the suitability of the pseudofermion representation for the study of the finite-energy spectral and dynamical properties [14, 15, 16]. The studies of Ref. [33] clarify the relation of the pseudofermion phase shifts and S matrices introduced in this paper to the corresponding quantities of the conventional spinon-holon scattering theory of Refs. [30, 31, 32]. The choice of scatterers and scattering centers of the latter theory is also based on associating the Bethe-ansatz quantum numbers with quantum objects, yet it does not profit from the invariances under the electron - rotated-electron unitary transformation. The clarification in this paper of the relation between the pseudofermion scattering mechanisms and symmetry provides further useful information about the PDT microscopic processes [15, 16].

Since the transport and dynamical properties [41] and other properties predicted by the 1D Hubbard model were observed in low-dimensional complex materials [42] and the investigations presented in Refs. [23, 24, 25] confirm that the PDT describes successfully the unusual finite-energy spectral features observed by angle-resolved photoelectron spectroscopy in quasi-1D organic metals, our results also contribute to the further understanding of the non-perturbative scattering mechanisms behind these properties. While the studies of this paper considered the 1D Hubbard model, our results are of general nature for many integrable interacting problems [43] and therefore have wide applicability.

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APPENDIX A: PSEUDOFORMION SCATTERING PROCESSES ENERGY CONSERVATION

Both the scattering phase shift $Q_{\alpha\nu}^{\Phi}(q_j)/2$, Eq. (4), and the overall phase shift $Q_{\alpha\nu}(q_j)/2 = Q_{\alpha\nu}^0/2 + Q_{\alpha\nu}^{\Phi}(q_j)/2$, Eq. (16), conserve the total energy.

Let us start by confirming that for $L \gg 1$ the above overall phase shift conserves the total energy. The $c0$ and $c1$ pseudofermion momentum distribution function deviations of Eq. (9) can be written as,

$$\Delta\mathcal{N}_{\alpha\nu}(\bar{q}_j) = \Delta\mathcal{N}_{\alpha\nu}^{(1)}(\bar{q}_j) + \Delta\mathcal{N}_{\alpha\nu}^{(2)}(\bar{q}_j); \quad \alpha\nu = c0, s1. \quad (\text{A1})$$

Here $\Delta\mathcal{N}_{\alpha\nu}(\bar{q}_j) \equiv \Delta N_{\alpha\nu}(q_j)$ and the deviations $\Delta\mathcal{N}_{\alpha\nu}^{(1)}(\bar{q}_j)$ and $\Delta\mathcal{N}_{\alpha\nu}^{(2)}(\bar{q}_j)$ are associated with the ground-state -

virtual-state transition and the virtual-state - “out”-state transition, respectively. The latter deviation reads,

$$\begin{aligned}\Delta\mathcal{N}_{\alpha\nu}^{(2)}(\bar{q}) &= \mathcal{N}_{\alpha\nu}^0(q + Q_{\alpha\nu}(q)/L) - \mathcal{N}_{\alpha\nu}^0(q) \\ &= \frac{1}{L} Q_{\alpha\nu}(q) \frac{\partial\mathcal{N}_{\alpha\nu}^0(\bar{q})}{\partial\bar{q}} = -\frac{1}{L} \text{sgn}(\bar{q}) \left[Q_{\alpha\nu}(\text{sgn}(\bar{q})q_{F\alpha\nu}^0) \right] \delta(q_{F\alpha\nu}^0 - |\bar{q}|); \quad \alpha = c0, s1. \end{aligned} \quad (\text{A2})$$

Here $q = \bar{q}$ is the virtual-state canonical momentum value, the *Fermi* momentum $q_{F\alpha\nu}^0$ of the $c0$ and $s1$ bands is given in Eq. (8), and $Q_{\alpha\nu}(q)/2$ is the phase shift (16). Use of Eq. (A2) in the energy spectrum (9) leads to the following energy spectrum to first order in the canonical-momentum distribution-function deviations,

$$\begin{aligned}\Delta E_{c0, s1}^{(2)} &= \sum_{\alpha\nu=c0, s1} \sum_{\bar{q}_j=-q_{\alpha\nu}^0}^{+q_{\alpha\nu}^0} \Delta\mathcal{N}_{\alpha\nu}^{(2)}(\bar{q}) \epsilon_{\alpha\nu}(\bar{q}) = -\frac{1}{L} \sum_{\iota=\pm 1} \sum_{\alpha\nu=c0, s1} \iota Q_{\alpha\nu}(\iota q_{F\alpha\nu}^0) \epsilon_{\alpha\nu}(q_{F\alpha\nu}^0) \\ &= -\frac{1}{L} \sum_{\iota=\pm 1} \sum_{\alpha\nu=c0, s1} \iota \left[Q_{\alpha\nu}^0 + Q_{\alpha\nu}^\Phi(\iota q_{F\alpha\nu}^0) \right] \epsilon_{\alpha\nu}(q_{F\alpha\nu}^0) = 0. \end{aligned} \quad (\text{A3})$$

In order to confirm that the factor $-\sum_{\iota=\pm 1} \sum_{\alpha\nu=c0, s1} \iota Q_{\alpha\nu}(\iota q_{F\alpha\nu}^0) \epsilon_{\alpha\nu}(q_{F\alpha\nu}^0)$, which multiplies $1/L$ in the last term on the right-hand side of Eq. (A3), vanishes we have used the symmetry $\epsilon_{\alpha\nu}(q) = \epsilon_{\alpha\nu}(-q)$ and Eq. (10) such that $\epsilon_{\alpha\nu}(q_{F\alpha\nu}^0) = 0$ for $\alpha\nu = c0, s1$. (We recall that the occupancies of other $\alpha\nu \neq c0, s1$ pseudofermion branches vanish for the ground state.) Thus, we have just found that the energy contribution of first order in the canonical-momentum distribution-function deviations originated by the collective excitation (2) vanishes.

Note that the energy (A3) decouples into two contributions, corresponding to the scatter-less phase shift $Q_{\alpha\nu}^0/2$ and scattering phase shift $Q_{\alpha\nu}^\Phi(\iota q_{F\alpha\nu}^0)/2$. This confirms that both these phase shifts conserve the energy independently.

Since the collective excitation (2) involves all virtual-state $c0$ and $s1$ pseudofermions, we used similar procedures for the evaluation of the energy contributions of order larger than one in the canonical-momentum distribution-function deviations. We find that all such contributions also vanish and decouple into independent and vanishing contributions corresponding to the scatter-less and scattering phase shifts. It follows that both the overall phase shift $Q_{\alpha\nu}(q_j)/2$ and the phase shifts $Q_{\alpha\nu}^0/2$ and $Q_{\alpha\nu}^\Phi(q_j)/2$ conserve the total energy.

APPENDIX B: ELEMENTARY TWO-PSEUDO-FERMION PHASE SHIFT EXPRESSIONS FOR $m \rightarrow 0$ AND $n \leq 1$ AND FOR $m \rightarrow 0$ AND $n \rightarrow 1$

We start by considering the limit $m \rightarrow 0$. The rapidity two-pseudofermion phase shifts $\pi \bar{\Phi}_{\alpha\nu, \alpha'\nu'}(r, r')$ are defined by the integral equations (A1)-(A13) of Ref. [6]. By Fourier transforming these equations after considering that $B = \infty$ and thus $r_s^0 = 4tB/U = \infty$ for finite values of U/t , we arrive to the following equations valid for $m \rightarrow 0$ and U/t finite,

$$\pi \bar{\Phi}_{c0, c0}(r, r') = -B(r - r') + \int_{-r_0}^{+r_0} dr'' A(r - r'') \pi \bar{\Phi}_{c0, c0}(r'', r'), \quad (\text{B1})$$

$$\pi \bar{\Phi}_{c0, s1}(r, r') = -\frac{1}{2} \arctan\left(\sinh\left(\frac{\pi}{2}(r - r')\right)\right) + \int_{-r_0}^{+r_0} dr'' A(r - r'') \pi \bar{\Phi}_{c0, s1}(r'', r'), \quad (\text{B2})$$

$$\begin{aligned}\pi \bar{\Phi}_{s1, c0}(r, r') &= -\frac{1}{2} \arctan\left(\sinh\left(\frac{\pi}{2}(r - r')\right)\right) + \frac{1}{4} \int_{-r_0}^{+r_0} dr'' \frac{\pi \bar{\Phi}_{c0, c0}(r'', r')}{\cosh\left(\frac{\pi}{2}(r - r'')\right)}; \quad r \neq \pm\infty \\ &= -\frac{\text{sgn}(r)\pi}{2\sqrt{2}}; \quad r = \pm\infty, \end{aligned} \quad (\text{B3})$$

$$\begin{aligned}\pi \bar{\Phi}_{s1, s1}(r, r') &= B(r - r') + \frac{1}{4} \int_{-r_0}^{+r_0} dr'' \frac{\pi \bar{\Phi}_{c0, s1}(r'', r')}{\cosh\left(\frac{\pi}{2}(r - r'')\right)}; \quad r \neq \pm\infty \\ &= \frac{\text{sgn}(r)\pi}{2\sqrt{2}}; \quad r = \pm\infty, \quad r' \neq r \\ &= [\text{sgn}(r)]\left(\frac{3}{2\sqrt{2}} - 1\right)\pi; \quad r = r' = \pm\infty, \end{aligned} \quad (\text{B4})$$

$$\pi \bar{\Phi}_{c0, c\nu}(r, r') = -\arctan\left(\frac{r-r'}{\nu}\right) + \int_{-r_0}^{+r_0} dr'' A(r-r'') \pi \bar{\Phi}_{c0, c\nu}(r'', r'); \quad \nu > 0, \quad (\text{B5})$$

$$\pi \bar{\Phi}_{c0, s\nu}(r, r') = 0; \quad \nu > 1, \quad (\text{B6})$$

$$\pi \bar{\Phi}_{s1, c\nu}(r, r') = \frac{1}{4} \int_{-r_0}^{+r_0} dr'' \frac{\pi \bar{\Phi}_{c0, c\nu}(r'', r')}{\cosh\left(\frac{\pi}{2}(r-r'')\right)}; \quad \nu > 0, \quad (\text{B7})$$

$$\begin{aligned} \pi \bar{\Phi}_{s1, s\nu}(r, r') &= \arctan\left(\frac{r-r'}{\nu-1}\right); \quad r \neq \pm\infty; \quad \nu > 1, \\ &= \pm \frac{\pi}{\sqrt{2}}; \quad r = \pm\infty; \quad \nu > 1, \end{aligned} \quad (\text{B8})$$

$$\pi \bar{\Phi}_{s\nu, c0}(r, r') = \pi \bar{\Phi}_{s\nu, c\nu'}(r, r') = 0; \quad \nu > 1, \quad (\text{B9})$$

$$\pi \bar{\Phi}_{s\nu, s1}(r, r') = \arctan\left(\frac{r-r'}{\nu-1}\right); \quad \nu > 1, \quad (\text{B10})$$

$$\pi \bar{\Phi}_{s\nu, s\nu'}(r, r') = \frac{1}{2} \Theta_{\nu, \nu'}(r-r') - \arctan\left(\frac{r-r'}{\nu+\nu'-2}\right) - \arctan\left(\frac{r-r'}{\nu+\nu'}\right), \quad \nu, \nu' > 1. \quad (\text{B11})$$

In the above expressions the function $\Theta_{\nu, \nu'}(x)$ is defined in Eq. (B.5) of Ref. [4],

$$B(r) = \int_0^\infty d\omega \frac{\sin(\omega r)}{\omega(1+e^{2\omega})} = \frac{i}{2} \ln \frac{\Gamma\left(\frac{1}{2} + i\frac{r}{4}\right) \Gamma\left(1 - i\frac{r}{4}\right)}{\Gamma\left(\frac{1}{2} - i\frac{r}{4}\right) \Gamma\left(1 + i\frac{r}{4}\right)}, \quad (\text{B12})$$

$$A(r) = \frac{1}{\pi} \frac{dB(r)}{dr} = \frac{1}{\pi} \int_0^\infty d\omega \frac{\cos(\omega r)}{1+e^{2\omega}}, \quad (\text{B13})$$

and

$$r_0 = \frac{4t \sin Q}{U}, \quad (\text{B14})$$

where Q is the parameter defined by Eq. (A.5) of Ref. [15] and $\Gamma(x)$ is the usual Γ function.

Moreover, the two-pseudofermion phase shifts,

$$\pi \bar{\Phi}_{c\nu, c0}(r, r') = \arctan\left(\frac{r-r'}{\nu}\right) - \frac{1}{\pi} \int_{-r_0}^{+r_0} dr'' \frac{\pi \bar{\Phi}_{c0, c0}(r'', r')}{\nu[1 + (\frac{r-r''}{\nu})^2]}; \quad \nu > 0, \quad (\text{B15})$$

$$\pi \bar{\Phi}_{c\nu, c\nu'}(r, r') = \frac{1}{2} \Theta_{\nu, \nu'}(r-r') - \frac{1}{\pi} \int_{-r_0}^{+r_0} dr'' \frac{\pi \bar{\Phi}_{c0, c\nu'}(r'', r')}{\nu[1 + (\frac{r-r''}{\nu})^2]}; \quad \nu, \nu' > 0, \quad (\text{B16})$$

$$\pi \bar{\Phi}_{c\nu, s\nu'}(r, r') = -\frac{1}{\pi} \int_{-r_0}^{+r_0} dr'' \frac{\pi \bar{\Phi}_{c0, s\nu'}(r'', r')}{\nu[1 + (\frac{r-r''}{\nu})^2]}; \quad \nu > 0, \quad (\text{B17})$$

remain as in Ref. [6]. Furthermore, we note that the four expressions (B1)-(B4) with $c0 = c$, $s1 = s$, $r = x$, and $r_0 = x_0$ are equivalent to expressions (A9)-(A12) of Ref. [39]. (For the phase shifts (B3) and (B4) this equality refers to values of r such that $r \neq \infty$.) The rapidity phase-shift expressions given here for $m \rightarrow 0$ correspond to some of the two-pseudofermion phase shifts plotted in units of π in Figs. 1-6.

Let us now consider the limit $m \rightarrow 0$ and $n \rightarrow 1$. Note that the expressions (B6) and (B8)-(B11) are valid for $m \rightarrow 0$ and all values of n such that $n \leq 1$. Thus, they also apply to the limit $n \rightarrow 1$. On the other hand, note that for all other rapidity two-pseudofermion phase-shifts the expressions for both $m \rightarrow 0$ and $n \rightarrow 1$ are obtained by considering $r_0 = 0$ in the above integral equations. Such a procedure leads to closed form analytical expressions for all rapidity two-pseudofermion phase shifts. Thus, for $m \rightarrow 0$ and $n \rightarrow 1$ we find,

$$\pi \bar{\Phi}_{c0, c0}(r, r') = -B(r - r'); \quad \pi \bar{\Phi}_{c0, s1}(r, r') = -\frac{1}{2} \arctan\left(\sinh\left(\frac{\pi}{2}(r - r')\right)\right), \quad (\text{B18})$$

$$\begin{aligned} \pi \bar{\Phi}_{s1, c0}(r, r') &= -\frac{1}{2} \arctan\left(\sinh\left(\frac{\pi}{2}(r - r')\right)\right); \quad r \neq \pm\infty \\ &= -\frac{\text{sgn}(r)\pi}{2\sqrt{2}}; \quad r = \pm\infty, \end{aligned} \quad (\text{B19})$$

$$\begin{aligned} \pi \bar{\Phi}_{s1, s1}(r, r') &= B(r - r'); \quad r \neq \pm\infty \\ &= \frac{\text{sgn}(r)\pi}{2\sqrt{2}}; \quad r = \pm\infty, \quad r' \neq r \\ &= [\text{sgn}(r)]\left(\frac{3}{2\sqrt{2}} - 1\right)\pi; \quad r = r' = \pm\infty, \end{aligned} \quad (\text{B20})$$

$$\pi \bar{\Phi}_{c0, c\nu}(r, r') = -\arctan\left(\frac{r - r'}{\nu}\right), \quad \pi \bar{\Phi}_{s1, c\nu}(r, r') = \pi \bar{\Phi}_{c0, s\nu'}(r, r') = 0, \quad \nu > 0, \quad \nu' > 1, \quad (\text{B21})$$

$$\begin{aligned} \pi \bar{\Phi}_{s1, s\nu}(r, r') &= \arctan\left(\frac{r - r'}{\nu - 1}\right); \quad r \neq \pm\infty; \quad \nu > 1, \\ &= \pm\frac{\pi}{\sqrt{2}}; \quad r = \pm\infty; \quad \nu > 1, \end{aligned} \quad (\text{B22})$$

$$\pi \bar{\Phi}_{c\nu, c0}(r, r') = \arctan\left(\frac{r - r'}{\nu}\right); \quad \pi \bar{\Phi}_{c\nu, c\nu'}(r, r') = \frac{1}{2} \Theta_{\nu, \nu'}(r - r'); \quad \pi \bar{\Phi}_{c\nu, s\nu'}(r, r') = 0, \quad \nu > 0, \quad (\text{B23})$$

and the phase-shift expressions (B9)-(B11) are n independent and then have the same expressions as for $n < 1$. The two-pseudofermion phase shift expressions derived here are used in the derivation of the equalities of Eq. (56).

APPENDIX C: THE TWO-PSEUDO-FERMION PHASE SHIFTS FOR $U/t \rightarrow 0$

Here we derive expressions for the $m \rightarrow 0$ two-pseudofermion phase shifts given in Appendix B that are plotted in Figs. 1-6 in units of π , $\pi \bar{\Phi}_{c0, c0}(q, q')$, $\pi \bar{\Phi}_{c0, s1}(q, q')$, $\pi \bar{\Phi}_{s1, c0}(q, q')$, $\pi \bar{\Phi}_{s1, s1}(q, q')$, $\pi \bar{\Phi}_{c0, c1}(q, q')$, and $\pi \bar{\Phi}_{s1, c1}(q, q')$, for the limit $U/t \rightarrow 0$. Moreover, we also provide large- U/t expressions for the rapidity functions involved in the evaluation of the two-pseudofermion phase-shift expansions (40)-(45).

The evaluation of the two-pseudofermion expressions provided here and of the expansions given in Eqs. (40)-(45) involves the use of Eq. (6) where the ground-state rapidity functions $\Lambda_{\alpha\nu}^0(q')$ are defined in terms of their inverse functions, whose expressions are given in Eqs. (A.1) and (A.2) of Ref. [15]. First, we use the latter expressions to derive the following closed-form expressions for the ground-state functions $k^0(q)$, $\Lambda_{c0}^0(q)$, $\Lambda_{c\nu}^0(q)$, and $\Lambda_{s1}^0(q)$, valid for zero spin density $m \rightarrow 0$, values of the electronic density $0 \leq n \leq 1$, and limiting on-site repulsion values $U/t \rightarrow 0$ and $U/t \gg 1$,

$$\begin{aligned} k^0(q) &= \frac{q}{2}; \quad |q| \leq 2k_F, \quad U/t \rightarrow 0 \\ &= \text{sgn}(q) [|q| - k_F]; \quad 2k_F \leq |q| < \pi/a, \quad U/t \rightarrow 0 \\ &= \text{sgn}(q) \pi; \quad |q| = \pi, \quad U/t \rightarrow 0 \\ &= q - \frac{4tn}{U} \ln(2) \sin(q); \quad |q| \leq \pi, \quad U/t \gg 1, \end{aligned} \quad (\text{C1})$$

$$\begin{aligned}
\Lambda_{c0}^0(q) &= \sin\left(\frac{q}{2}\right); \quad |q| \leq 2k_F, \quad U/t \rightarrow 0 \\
&= \text{sgn}(q) \sin\left(|q| - k_F\right); \quad 2k_F \leq |q| < \pi, \quad U/t \rightarrow 0 \\
&= 0; \quad |q| = \pi, \quad U/t \rightarrow 0 \\
&= \sin(q) - \frac{2tn}{U} \ln(2) \sin(2q); \quad |q| \leq \pi, \quad U/t \gg 1,
\end{aligned} \tag{C2}$$

$$\begin{aligned}
\Lambda_{c\nu}^0(q) &= \text{sgn}(q) \sin\left(\frac{(|q| + \pi n)}{2}\right); \quad 0 < |q| < (\pi - 2k_F), \quad U/t \rightarrow 0 \\
&= 0; \quad q = 0, \quad U/t \rightarrow 0 \\
&= \pm\infty; \quad q = \pm(\pi - 2k_F), \quad U/t \rightarrow 0 \\
&= \frac{\nu U}{4t} \tan\left(\frac{q}{2\delta}\right); \quad 0 \leq |q| \leq (\pi - 2k_F), \quad U/t \gg 1,
\end{aligned} \tag{C3}$$

for $\nu > 0$, and

$$\begin{aligned}
\Lambda_{s1}^0(q) &= \sin(q); \quad |q| < k_F, \quad U/t \rightarrow 0 \\
&= \pm\infty; \quad q = \pm k_F, \quad U/t \rightarrow 0 \\
&= \frac{U}{2\pi t} \text{arcsinh}\left(\tan\left(\frac{q}{n}\right)\right); \quad |q| \leq k_F, \quad U/t \gg 1,
\end{aligned} \tag{C4}$$

respectively.

Next, we use Eqs. (C1)-(C4) in the integral equations given in the Appendix A of Ref. [6], which define the two-pseudofermion phase shifts. (Such equations were also presented in Ref. [44] with a slightly different notation.) By manipulation of these equations for the limit $U/t \rightarrow 0$, we find the expressions for the above bare-momentum two-pseudofermion phase shifts in units of π given below. For the phase shifts $\pi \Phi_{s1, \alpha\nu}(q, q')$ we provide the values for $U/t = 0$ and $U/t \rightarrow 0$, when different. The expressions read,

$$\pi \Phi_{c0, c0}(q, q') = -\text{sgn}\left(\sin k_{c0}^0(q) - \sin k_{c0}^0(q')\right) \frac{\pi}{C_c(q)} + \delta_{|q|, 2k_F} \delta_{q, q'} [\text{sgn}(q)] \left(\frac{3}{2\sqrt{2}} - 1\right) \pi, \tag{C5}$$

$$\pi \Phi_{c0, s1}(q, q') = -\text{sgn}\left(\sin k_{c0}^0(q) - c_{s1}(q') \sin(q')\right) \frac{\pi}{C_c(q)}, \tag{C6}$$

$$\begin{aligned}
\pi \Phi_{s1, c0}(q, q') &= -\text{sgn}\left(\sin(q) - \sin k_{c0}^0(q')\right) \frac{\pi}{2}; \quad q \neq \pm k_F \\
&= -\frac{\text{sgn}(q)\pi}{2\sqrt{2}}; \quad q = \pm k_F; \quad U/t \rightarrow 0 \\
&= 0; \quad q = \pm k_F; \quad U/t = 0,
\end{aligned} \tag{C7}$$

$$\begin{aligned}
\pi \Phi_{s1, s1}(q, q') &= 0; \quad q \neq \pm k_F \\
&= \frac{\text{sgn}(q)\pi}{2\sqrt{2}} \left[1 + \delta_{q, q'} 2(1 - \sqrt{2})\right]; \quad q = \pm k_F; \quad U/t \rightarrow 0 \\
&= 0; \quad q = \pm k_F; \quad U/t = 0,
\end{aligned} \tag{C8}$$

$$\pi \Phi_{c0, c1}(q, q') = -\text{sgn}\left(\sin k_{c0}^0(q) - c_{c1}(q') \sin k_{c1}^0(q')\right) \frac{2\pi}{C_c(q)}, \tag{C9}$$

and

$$\pi \Phi_{s1, c1}(q, q') = -\theta(k_F - |q|) \text{sgn}\left(\sin q - \sin k_{c1}^0(q')\right) \frac{\pi}{2}, \tag{C10}$$

respectively. Here the sign function is such that $\text{sgn}(0) = 0$ and $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \leq 0$ and thus $\pi \Phi_{s1, c1}(\pm k_F, q') = 0$. In the above equations, $k_{c0}^0(q) = \lim_{U/t \rightarrow 0} k^0(q)$ where the $U/t \rightarrow 0$ value of $k^0(q)$ is given in Eq. (C1),

$$\begin{aligned} k_{c1}^0(q) &= \frac{q}{2} + \text{sgn}(q) k_F; \quad 0 < |q| \leq [\pi - 2k_F] \\ &= 0; \quad q = 0, \end{aligned} \quad (\text{C11})$$

$$C_c(q) = 2 \left[\theta(2k_F - |q|) + \sqrt{2} \delta_{|q|, 2k_F} + 2\theta(\pi - |q|)\theta(|q| - 2k_F) + \delta_{|q|, \pi} \right], \quad (\text{C12})$$

and

$$\begin{aligned} c_{s1}(q) &= 1, \quad |q| < k_F; \quad c_{s1}(q) = \infty, \quad q = \pm k_F \\ c_{c1}(q) &= 1, \quad |q| < [\pi - 2k_F]; \quad c_{c1}(q) = \infty, \quad q = \pm[\pi - 2k_F]. \end{aligned} \quad (\text{C13})$$

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- [1] Elliott H. Lieb, F. Y. Wu, Phys. Rev. Lett. 20 (1968) 1445.
 - [2] M. Takahashi, Prog. Theor. Phys. 47 (1972) 69.
 - [3] P. B. Ramos, M. J. Martins, J. Phys. A 30 (1997) L195; M. J. Martins, P.B. Ramos, Nucl. Phys. B 522 (1998) 413.
 - [4] J. M. P. Carmelo, J. M. Román, K. Penc, Nucl. Phys. B 683 (2004) 387.
 - [5] J. M. P. Carmelo, P. D. Sacramento, Phys. Rev. B 68 (2003) 085104.
 - [6] J. M. P. Carmelo, cond-mat/0405411.
 - [7] F. Woynarovich, J. Phys. A: Math. Gen. 22 (1989) 4243.
 - [8] M. Ogata, H. Shiba, Phys. Rev. B 41(1990) 2326; M. Ogata, T. Sugiyama, H. Shiba, Phys. Rev. B 43 (1991) 8401.
 - [9] N. Kawakami, S. K. Yang, Phys. Lett. A 148 (1990) 359.
 - [10] H. Frahm, V. E. Korepin, Phys. Rev. B 42 (1990) 10 553; H. Frahm, V. E. Korepin, Phys. Rev. B 43 (1991) 5653.
 - [11] M. Brech, J. Voit, H. Buttner, Europhys. Lett. 12 (1990) 289.
 - [12] J. M. P. Carmelo, A. H. Castro Neto, Phys. Rev. Lett. 70 (1993) 1904; J. M. P. Carmelo, A. H. Castro Neto, D. K. Campbell, Phys. Rev. B 50 (1994) 3667; J. M. P. Carmelo, A. H. Castro Neto, D. K. Campbell, Phys. Rev. B 50 (1994) 3683.
 - [13] K. Penc, J. Sólyom, Phys. Rev. B 47 (1993) 6273 (1993).
 - [14] J. M. P. Carmelo, K. Penc, cond-mat/0311075.
 - [15] J. M. P. Carmelo, K. Penc, D. Bozi, Nucl. Phys. B 725 (2005) 421; J. M. P. Carmelo, K. Penc, D. Bozi, Nucl. Phys. B 737 (2006) 351, Erratum.
 - [16] J. M. P. Carmelo, L. M. Martelo, K. Penc, Nucl. Phys. B 737 (2006) 237; J. M. P. Carmelo, K. Penc, Phys. Rev. B 73 (2006) at press.
 - [17] A. Brooks Harris, Robert V. Lange, Phys. Rev. 157 (1967) 295; A. H. MacDonald, S. M. Girvin, D. Yoshioka, Phys. Rev. B 37 (1988) 9753.
 - [18] O. J. Heilmann, E. H. Lieb, Ann. N. Y. Acad. Sci. 172 (1971) 583; E. H. Lieb, Phys. Rev. Lett. 62 (1989) 1201.
 - [19] C. N. Yang, Phys. Rev. Lett. 63 (1989) 2144.
 - [20] Karlo Penc, Karen Hallberg, Frédéric Mila, Hiroyuki Shiba, Phys. Rev. Lett. 77 (1996) 1390; Karlo Penc, Karen Hallberg, Frédéric Mila, Hiroyuki Shiba, Phys. Rev. B 55 (1997) 15 475.
 - [21] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, Nucl. Phys. B 241 (1984) 333.
 - [22] H. J. Schulz, Phys. Rev. Lett. 64 (1990) 2831; J. M. P. Carmelo, A. H. Castro Neto, D. K. Campbell, Phys. Rev. Lett. 73 (1994) 926 and Erratum 74 (1995) 3089.
 - [23] M. Sing, U. Schwingenschlögl, R. Claessen, P. Blaha, J. M. P. Carmelo, L. M. Martelo, P. D. Sacramento, M. Dressel, C. S. Jacobsen, Phys. Rev. B 68 (2003) 125111.
 - [24] J. M. P. Carmelo, K. Penc, L. M. Martelo, P. D. Sacramento, J. M. B. Lopes dos Santos, R. Claessen, M. Sing, U. Schwingenschlögl, Europhys. Lett. 67 (2004) 233; J. M. P. Carmelo, D. Bozi, P. D. Sacramento, K. Penc, submitted for publication.
 - [25] J. M. P. Carmelo, F. Guinea, K. Penc, P. D. Sacramento, Europhys. Lett. 68 (2004) 839.
 - [26] H. Benthien, F. Gebhard, E. Jeckelmann, Phys. Rev. Lett. 92 (2004) 256401.
 - [27] D. Jaksch, P. Zoller, Ann. Phys. 315 (2005) 52.
 - [28] Y. Chen, private communication.
 - [29] J. M. P. Carmelo, J. Phys.: Cond. Mat. 17 (2005) 5517.
 - [30] N. Andrei, *Series on Modern Condensed Matter Physics - Vol. 6*, 458, World Scientific, Lecture Notes of ICTP Summer Course, Editors: S. Lundquist, G. Morandi, Yu Lu [cond-mat/9408101].
 - [31] Fabian H. L. Essler, Vladimir E. Korepin, Phys. Rev. Lett. 72 (1994) 908.

- [32] Fabian H. L. Essler, Vladimir E. Korepin, Nucl. Phys. B 426 (1994) 505.
- [33] J. M. P. Carmelo, K. E. Hibberd, N. Andrei, cond-mat/0603446.
- [34] John R. Taylor, *Scattering theory: the quantum theory of nonrelativistic collisions* (Robert E. Krieger Publishing Company, Malabar, Florida, 1987).
- [35] For a basic introduction to chromodynamics see Martinus Veltman, *Facts and Mysteries in Elementary Particle Physics* (World Scientific, New Jersey, 2003).
- [36] J. M. P. Carmelo, D. Bozi, unpublished.
- [37] Gerald D. Mahan, *Many-particle physics* (Kluwer Academic/Plenum Publishers, New York, 2000), Chapter 4; F. G. Fumi, Philos. Mag. 46 (1955) 1007.
- [38] J. Carmelo and A. A. Ovchinnikov, J. Phys.: Condens. Matter **3**, 757 (1991).
- [39] J. M. P. Carmelo, P. Horsch, A. A. Ovchinnikov, Phys. Rev. B 45 (1992) 7899.
- [40] H. C. Ohanian, *Principles of quantum mechanics* (Prentice Hall, Englewood Cliffs, New Jersey, 1993).
- [41] N. M. R. Peres, J. M. P. Carmelo, D. K. Campbell, A. W. Sandvik, Z. Phys. B 103 (1997) 217; J. M. P. Carmelo, P. Horsch, D. K. Campbell, A. H. Castro Neto, Phys. Rev. B (RC) 48 (1993) 4200.
- [42] Dionys Baeriswyl, José Carmelo, Kazumi Maki, Synth. Met. 21 (1987) 271. J. M. P. Carmelo, N. M. R. Peres, P. D. Sacramento, Phys. Rev. Lett. 84 (2000) 4673; D. Controzzi, F.H.L. Essler, A.M. Tsvelik, Phys. Rev. Lett. 86 (2001) 680; H. Kishida, M. Ono, K. Miura, H. Okamoto, M. Izumi, T. Manako, M. Kawasaki, Y. Taguchi, Y. Tokura, T. Tohyama, K. Tsutsui, S. Maekawa, Phys. Rev. Lett. 87 (2001) 177401.
- [43] H. Bethe, Z. Phys. 71 (1931) 205; Elliott H. Lieb, Werner Liniger, Phys. Rev. 130 (1963) 1605; N. Andrei, J. Lowenstein, Phys. Rev. Lett. 43 (1979) 1698; N. Andrei, Phys. Rev. Lett. 45 (1980) 379; P.- A. Bares, J. M. P. Carmelo, J. Ferrer, P. Horsch, Phys. Rev. B 46 (1992) 14624.
- [44] J. M. P. Carmelo, N. M. R. Peres, Phys. Rev. B 56 (1997) 3717.